ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$n\in\{2,3,\dots\}\ \Rightarrow\ \sqrt[n+1]{1+(n+1)\sqrt{\binom{2n}{n}\left(2^2+\left(\frac{2^2}{2}\right)^2+\left(\frac{2^3}{3}\right)^2+\dots+\left(\frac{2^{n+1}}{n+1}\right)^2\right)}}>3$$

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Since $\binom{2n}{n}=\binom{n}{0}^2+\binom{n}{1}^2+\dots+\binom{n}{n}^2$, then by the Cauchy – Schwarz inequality, we have

$$\sqrt{\binom{2n}{n}\left(2^2+\left(\frac{2^2}{2}\right)^2+\left(\frac{2^3}{3}\right)^2+\dots+\left(\frac{2^{n+1}}{n+1}\right)^2\right)}=\sqrt{\sum_{k=0}^{n}\binom{n}{k}^2\cdot\sum_{k=0}^{n}\left(\frac{2^{k+1}}{k+1}\right)^2}$$

$$\ge\sum_{k=0}^{n}2^{k+1}.\frac{\binom{n}{k}}{k+1}$$

$$=\sum_{k=0}^{n}2^{k+1}.\frac{\binom{n+1}{k+1}}{n+1}=\frac{1}{n+1}.\sum_{k=1}^{n+1}\binom{n+1}{k}.2^k.1^{n+1-k}=\frac{(2+1)^{n+1}-1}{n+1}=\frac{3^{n+1}-1}{n+1},$$

with equality if $\frac{k+1}{2^{k+1}}\binom{n}{k}=\text{constant}$, $\forall k\in\{0,1,\dots,n\}$, which is not true for any $n\ge 2$.

Therefore

$$\sqrt[n+1]{1+(n+1)\sqrt{\binom{2n}{n}\left(2^2+\left(\frac{2^2}{2}\right)^2+\left(\frac{2^3}{3}\right)^2+\dots+\left(\frac{2^{n+1}}{n+1}\right)^2\right)}}>3$$