ROMANIAN MATHEMATICAL MAGAZINE

If $\left(a > 0 \text{ and } \mathrm{A} = a^2 + a^4 + \frac{1}{a^6}\right)$, then :

$$1620\mathrm{A} \leq 3645 + 5\left(\frac{189 + 5\mathrm{A}^3}{108}\right)^5$$

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$$\mathrm{A} \overset{?}{\leq} \frac{189 + 5\mathrm{A}^3}{108} \Leftrightarrow 5\mathrm{A}^3 - 108\mathrm{A} + 189 \overset{?}{\geq} 0$$

$$\Leftrightarrow (\mathrm{A} - 3)\big((\mathrm{A} - 3)(5\mathrm{A} + 30) + 27\big) \overset{?}{\geq} 0 \to \text{true} \because \mathrm{A} = a^2 + a^4 + \frac{1}{a^6}$$

$$\overset{\mathrm{A-G}}{\geq} 3\sqrt[3]{a^2 . a^4 . \frac{1}{a^6}} = 3 \therefore 1620\mathrm{A} - \left(3645 + 5\left(\frac{189 + 5\mathrm{A}^3}{108}\right)^5\right)$$

$$\leq 1620\left(\frac{189 + 5\mathrm{A}^3}{108}\right) - \left(3645 + 5\left(\frac{189 + 5\mathrm{A}^3}{108}\right)^5\right) \overset{?}{\leq} 0$$

$$\Leftrightarrow \mathrm{t}^5 - 324\mathrm{t} + 729 \overset{?}{\geq} 0 \quad \left(\mathrm{t} = \frac{189 + 5\mathrm{A}^3}{108}\right)$$

$$\Leftrightarrow (\mathrm{t} - 3)\big((\mathrm{t} - 3)(\mathrm{t}^3 + 6\mathrm{t}^2 + 27\mathrm{t} + 108) + 81\big) \overset{?}{\geq} 0 \to \text{true} \because \mathrm{t} = \frac{189 + 5\mathrm{A}^3}{108} \geq \mathrm{A}$$

$$= a^2 + a^4 + \frac{1}{a^6} \overset{\mathrm{A-G}}{\geq} 3\sqrt[3]{a^2 . a^4 . \frac{1}{a^6}} = 3$$

$$\therefore 1620\mathrm{A} \leq 3645 + 5\left(\frac{189 + 5\mathrm{A}^3}{108}\right)^5, '' ='' \text{ iff } a = 1 \text{ (QED)}$$