ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ *such that* $abc = 1$, then prove that :

$$\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c} \geq \sqrt[3]{24+\sqrt[3]{27+\sqrt{\det\begin{pmatrix} \frac{2}{a}-a^2 & c^2 & b^2 \\ c^2 & \frac{2}{b}-b^2 & a^2 \\ b^2 & a^2 & \frac{2}{c}-c^2 \end{pmatrix}}}}$$

Proposed by Pavlos Trifon-Greece

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\det\begin{pmatrix} \frac{2}{a}-a^2 & c^2 & b^2 \\ c^2 & \frac{2}{b}-b^2 & a^2 \\ b^2 & a^2 & \frac{2}{c}-c^2 \end{pmatrix} \overset{abc=1}{=}$$

$$(2bc-a^2)\left((2ca-b^2)(2ab-c^2)-a^4\right)+c^2\left(a^2b^2-c^2(2ab-c^2)\right)$$
$$+b^2\left(c^2a^2-b^2(2ca-b^2)\right)$$
$$= a^6+b^6+c^6+2a^3b^3+2b^3c^3+2c^3a^3+9a^2b^2c^2-6abc(a^3+b^3+c^3)$$
$$= \left(\sum_{cyc} a^3\right)^2 + 9a^2b^2c^2 - 6abc\left(\sum_{cyc} a^3\right) = \left(\sum_{cyc} a^3 - 3abc\right)^2$$

$$\therefore \sqrt[3]{24+\sqrt[3]{27+\sqrt{\det\begin{pmatrix} \frac{2}{a}-a^2 & c^2 & b^2 \\ c^2 & \frac{2}{b}-b^2 & a^2 \\ b^2 & a^2 & \frac{2}{c}-c^2 \end{pmatrix}}}} =$$

$$\sqrt[3]{24+\sqrt[3]{27+\sqrt{\left(\sum_{cyc} a^3 - 3abc\right)^2}}} \overset{abc=1}{=} \sqrt[3]{24+\sqrt[3]{\sum_{cyc} a^3 + 24abc}}$$

$$\overset{?}{\leq} \sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c} \Leftrightarrow \sum_{cyc} x \overset{?}{\geq} \sqrt[3]{24xyz+\sqrt[3]{\sum_{cyc} x^9 + 24x^3y^3z^3}}$$

$$\left(x=\sqrt[3]{a}, y=\sqrt[3]{b}, z=\sqrt[3]{c} \text{ and } xyz=1 \because abc = 1\right)$$

$$\Leftrightarrow \left(\sum_{cyc} x\right)^3 - 24xyz \overset{?}{\geq} \sqrt[3]{\sum_{cyc} x^9 + 24x^3y^3z^3}$$

$$\Leftrightarrow \left(\left(\sum_{cyc} x\right)^3 - 24xyz\right)^3 \underset{(\bullet)}{\overset{?}{\geq}} \sum_{cyc} x^9 + 24x^3y^3z^3$$

Assigning $y + z = X, z + x = Y, x + y = Z \Rightarrow X + Y - Z = 2z > 0, Y + Z - X = 2x > 0$ and $Z + X - Y = 2y > 0 \Rightarrow X + Y > Z, Y + Z > X, Z + X > Y$
$\Rightarrow$ X, Y, Z form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} x = \sum_{cyc} X = 2s \Rightarrow \sum_{cyc} x = s \to (1)$
$\Rightarrow x = s - X, y = s - Y, z = s - Z$ and such substitutions $\Rightarrow$
$\sum_{cyc} xy = \sum_{cyc}(s - X)(s - Y) \Rightarrow \sum_{cyc} xy = 4Rr + r^2 \to (2)$ and

$$\sum_{cyc} x^3 = \left(\sum_{cyc} x\right)^3 - 3\prod_{cyc}(y + z) \overset{via\ (1)}{=} s^3 - 3XYZ = s^3 - 12Rrs$$

$$\Rightarrow \sum_{cyc} x^3 = s^3 - 12Rrs \to (3) \text{ and also}, \sum_{cyc} x^3y^3 = \left(\sum_{cyc} xy\right)^3 - 3xyz\prod_{cyc}(y + z)$$

$$\overset{via\ (2)\ and\ (3)}{=} \left(4Rr + r^2\right)^3 - 3r^2s.4Rrs \Rightarrow \sum_{cyc} x^3y^3 = \left(4Rr + r^2\right)^3 - 12Rr^3s^2 \to (4)$$

$$\therefore \sum_{cyc} x^9 = \left(\sum_{cyc} x^3\right)^3 - 3\left(x^3 + y^3\right)\left(y^3 + z^3\right)\left(z^3 + x^3\right)$$

$$= \left(\sum_{cyc} x^3\right)^3 - 3\left(\left(\sum_{cyc} x^3\right)\left(\sum_{cyc} x^3y^3\right) - x^3y^3z^3\right)$$

$$\therefore (*) \Leftrightarrow \left(\left(\sum_{cyc} x\right)^3 - 24xyz\right)^3 \overset{?}{\geq}$$

$$\left(\sum_{cyc} x^3\right)^3 - 3\left(\left(\sum_{cyc} x^3\right)\left(\sum_{cyc} x^3y^3\right) - x^3y^3z^3\right) + 24x^3y^3z^3$$

$$\overset{via\ (1),(3)\ and\ (4)}{\Leftrightarrow} \left(s^3 - 24r^2s\right)^3 \overset{?}{\geq} \left(s^3 - 12Rrs\right)^3 + 27r^6s^3$$

$$-3\left(s^3 - 12Rrs\right)\left(\left(4Rr + r^2\right)^3 - 12Rr^3s^2\right)$$

$$\Leftrightarrow (3R - 6r)s^6 - r\left(36R^2 + 3Rr - 144r^2\right)s^4$$

$$+r^2s^2\left(160R^3 + 48R^2r + 3Rr^2 - 1154r^3\right) - 3Rr^3(4R + r)^3 \underset{(*)}{\overset{?}{\geq}} 0 \text{ and}$$

$$\because (3R-6r)\left(s^2-16Rr+5r^2\right)^3 \overset{\text{Gerretsen + Euler}}{\geq} 0 \therefore \text{ in order to prove } (*),$$

$$\text{it suffices to prove : LHS of } (*) \geq (3R-6r)\left(s^2-16Rr+5r^2\right)^3$$

$$\Leftrightarrow (108R-120r)(R-2r)s^4-6r^2s^4+rs^2\left(160R^3+48R^2r+3Rr^2-1154r^3\right)$$

$$-3Rr^2(4R+r)^3 \underbrace{\overset{?}{\geq}}_{(**)} 0$$

$$\text{Now, LHS of } (**) \overset{\text{Gerretsen}}{\geq} (108R-120r)(R-2r)\left(16Rr-5r^2\right)s^2$$

$$-6r^2s^2\left(4R^2+4Rr+3r^2\right)+rs^2\left(160R^3+48R^2r+3Rr^2-1154r^3\right)$$

$$-3Rr^2(4R+r)^3 \overset{?}{\geq} 0$$

$$\Leftrightarrow \left(1888R^3-5892R^2r+5499Rr^2-2372r^3\right)s^2 \underbrace{\overset{?}{\geq}}_{(***)} 3Rr(4R+r)^3$$

$$\because 1888R^3-5892R^2r+5499Rr^2-2372r^3$$

$$=(R-2r)\left(1888R^2-2116Rr+1267r^2\right)+162r^3 \overset{\text{Euler}}{\geq} 162r^3>0$$

$$\therefore \text{LHS of } (***) \overset{\text{Gerretsen}}{\geq}$$

$$\left(1888R^3-5892R^2r+5499Rr^2-2372r^3\right)\left(16Rr-5r^2\right) \overset{?}{\geq} 3Rr(4R+r)^3$$

$$\Leftrightarrow 30016t^4-103856t^3+117408t^2-65450t+11860 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)\left(30016t^2+16208t+62176\right)+118422\right) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \Rightarrow (\bullet) \text{ is true}$$

$$\therefore \sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c} \geq \sqrt[3]{24+\sqrt[3]{27+\sqrt{\det\begin{pmatrix} \frac{2}{a}-a^2 & c^2 & b^2 \\ c^2 & \frac{2}{b}-b^2 & a^2 \\ b^2 & a^2 & \frac{2}{c}-c^2 \end{pmatrix}}}}$$

$$\forall\, a,b,c>0 \,\middle|\, abc=1,''='' \text{ iff } a=b=c \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have

$$\det\begin{pmatrix} \frac{2}{a}-a^2 & c^2 & b^2 \\ c^2 & \frac{2}{b}-b^2 & a^2 \\ b^2 & a^2 & \frac{2}{c}-c^2 \end{pmatrix} = \det\begin{pmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{pmatrix}$$

$$=(2bc-a^2)[(2ca-b^2)(2ab-c^2)-a^4]-c^2[c^2(2ab-c^2)-a^2b^2]$$

$$+b^2[c^2a^2-b^2(2ca-b^2)]$$

$$=a^6+b^6+c^6+2(a^3b^3+b^3c^3+c^3a^3)-6abc(a^3+b^3+c^3)+9(abc)^2$$

$$=(a^3+b^3+c^3-3abc)^2=(a^3+b^3+c^3-3)^2.$$

So it suffices to prove that

$$\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\geq\sqrt[3]{24+\sqrt[3]{24+a^3+b^3+c^3}}.$$

By AM − GM inequality, we have

$$\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}=\sqrt[3]{a+b+c+3\left(\sqrt[3]{a}+\sqrt[3]{b}\right)\left(\sqrt[3]{b}+\sqrt[3]{c}\right)\left(\sqrt[3]{c}+\sqrt[3]{a}\right)}$$

$$\geq\sqrt[3]{a+b+c+3.8\sqrt[3]{abc}}=\sqrt[3]{\sqrt[3]{a^3+b^3+c^3+3(a+b)(b+c)(c+a)}+24}$$

$$\geq\sqrt[3]{\sqrt[3]{a^3+b^3+c^3+3.8abc}+24}=\sqrt[3]{24+\sqrt[3]{24+a^3+b^3+c^3}},$$

so the proof is complete. Equality holds iff $a=b=c=1$.