ROMANIAN MATHEMATICAL MAGAZINE

If $x > 0$, then :

$$\pi\left(1 + (\arctan x)(\operatorname{arccot} x)^{-1}\right) > 8 + 2\left(\ln^2 \arctan x + \ln^2 \operatorname{arccot} x\right)$$

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$$\text{Let } F(x) = \pi\left(1 + (\arctan x)(\operatorname{arccot} x)^{-1}\right) - 8$$
$$-2\left(\ln^2 \arctan x + \ln^2 \operatorname{arccot} x\right) \ \forall\, x > 0$$
$$\therefore F'(x) =$$
$$-\frac{4(\operatorname{arccot} x)^2(\arctan x)\ln(\arctan x) - 4(\arctan x)^2(\operatorname{arccot} x)\ln(\operatorname{arccot} x) + \pi\begin{pmatrix}\operatorname{arccot} x\\ -\arctan x\end{pmatrix}}{(x^2+1)(\arctan x)^2(\operatorname{arccot} x)^2}$$
$$\frac{4(\arctan x)^2(\operatorname{arccot} x)\ln(\operatorname{arccot} x) - 4(\operatorname{arccot} x)^2(\arctan x)\ln(\arctan x) - \pi\begin{pmatrix}\operatorname{arccot} x\\ -\arctan x\end{pmatrix}}{(\arctan x)^2(\operatorname{arccot} x)^2}$$
$$= 4\left(\frac{\ln(\operatorname{arccot} x)}{\operatorname{arccot} x} - \frac{\ln(\arctan x)}{\arctan x} - \frac{\pi}{4}\frac{(\operatorname{arccot} x - \arctan x)}{(\arctan x)^2(\operatorname{arccot} x)^2}\right)$$
$$= 4\left(\frac{\ln(\operatorname{arccot} x)}{\operatorname{arccot} x} - \frac{\ln(\arctan x)}{\arctan x} - \frac{1}{2}\frac{(\operatorname{arccot} x + \arctan x)(\operatorname{arccot} x - \arctan x)}{(\arctan x)^2(\operatorname{arccot} x)^2}\right)$$
$$= 4\left(\frac{\ln(\operatorname{arccot} x)}{\operatorname{arccot} x} - \frac{\ln(\arctan x)}{\arctan x} - \frac{1}{2}\left(\frac{1}{(\arctan x)^2} - \frac{1}{(\operatorname{arccot} x)^2}\right)\right)$$
$$= 4\left(\left(\frac{1}{2(\operatorname{arccot} x)^2} + \frac{\ln(\operatorname{arccot} x)}{\operatorname{arccot} x}\right) - \left(\frac{1}{2(\arctan x)^2} + \frac{\ln(\arctan x)}{\arctan x}\right)\right)$$
$$\Rightarrow F'(x) = \frac{4}{x^2+1}\cdot\left(\begin{matrix}\left(\dfrac{1}{2(\operatorname{arccot} x)^2} + \dfrac{\ln(\operatorname{arccot} x)}{\operatorname{arccot} x}\right)\\ -\left(\dfrac{1}{2(\arctan x)^2} + \dfrac{\ln(\arctan x)}{\arctan x}\right)\end{matrix}\right)$$
$$\text{Let } f(m) = \frac{1}{2m^2} + \frac{\ln m}{m}\ \forall\, m > 0 \therefore f'(m) = \frac{m - 1 - m\ln m}{m^3} = \frac{m-1}{m^3} - \frac{\ln m}{m^2}$$
$$\le \frac{m-1}{m^3} - \frac{1 - \frac{1}{m}}{m^2}\left(\because \ln m \ge 1 - \frac{1}{m}\ \forall\, m > 0\right) = 0 \therefore \frac{m - 1 - m\ln m}{m^3} \le 0 \to (1)$$

$\boxed{\textit{Case 1}}$ $0 < x \le 1$ *and then* : $\operatorname{arccot} x \ge \arctan x$ *and* $F'(x)$

$$= \frac{4}{x^2+1}\cdot\left(f(\operatorname{arccot} x) - f(\arctan x)\right) \overset{\text{MVT}}{=}$$
$$\boxed{\frac{4}{x^2+1}\cdot(\operatorname{arccot} x - \arctan x)\cdot\frac{\xi - 1 - \xi\ln\xi}{\xi^3}} \overset{\text{via (1)}}{\le} 0$$
$$\Rightarrow F(x) \text{ is} \downarrow \text{on } (0,1] \Rightarrow F(x) \ge F(1) = \pi\left(1 + \frac{16}{\pi^2}\right) - 8 - 4\ln^2\frac{\pi}{4} \approx .0011 > 0$$
$$\Rightarrow \pi\left(1 + (\arctan x)(\operatorname{arccot} x)^{-1}\right) > 8 + 2\left(\ln^2 \arctan x + \ln^2 \operatorname{arccot} x\right)$$

$\boxed{\textit{Case 2}}$ $x \ge 1$ *and then* : $\arctan x \ge \operatorname{arccot} x$ *and* $F'(x)$

$$= -\frac{4}{x^2+1}.\left(f(\arctan x) - f(\operatorname{arccot} x)\right) \overset{MVT}{=}$$

$$\boxed{-\frac{4}{x^2+1}.(\arctan x - \operatorname{arccot} x).\frac{\xi - 1 - \xi \ln \xi}{\xi^3}} \overset{via\ (1)}{\geq} 0$$

$$\Rightarrow F(x) \text{ is } \uparrow \text{ on } [1,\infty) \Rightarrow F(x) \geq F(1) = \pi\left(1 + \frac{16}{\pi^2}\right) - 8 - 4\ln^2\frac{\pi}{4} \approx .0011 > 0$$

$$\Rightarrow \pi\left(1 + (\arctan x)(\operatorname{arccot} x)^{-1}\right) > 8 + 2\left(\ln^2 \arctan x + \ln^2 \operatorname{arccot} x\right)$$

$$\therefore \text{combining both cases}, \pi\left(1 + (\arctan x)(\operatorname{arccot} x)^{-1}\right)$$

$$> 8 + 2\left(\ln^2 \arctan x + \ln^2 \operatorname{arccot} x\right) \ \forall\, x > 0 \ (QED)$$