ROMANIAN MATHEMATICAL MAGAZINE

Find x, y, z and $t \in Z^{+}$:

$$\frac{y+z+t}{x}+\frac{z+t}{x+y}+\frac{t}{x+y+z}=x+y+z+t-4$$

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$$\left(\frac{y+z+t}{x}+1\right)+\left(\frac{z+t}{x+y}+1\right)+\left(\frac{t}{x+y+z}+1\right)=x+y+z+t-1$$

$$\frac{y+x+z+t}{x}+\frac{y+x+z+t}{x+y}+\frac{y+x+z+t}{x+y+z}=x+y+z+t-1$$

$$x+y+z+t>0$$

$$\frac{1}{x}+\frac{1}{x+y}+\frac{1}{x+y+z}+\frac{1}{x+y+z+t}=1$$

$$1)\ case\ x=1\ cannot\ be\ solved:$$

$$\frac{1}{x}>\frac{1}{x+y}>\frac{1}{x+y+z}>\frac{1}{x+y+z+t}\rightarrow\frac{4}{x}>1\rightarrow\boxed{x<4}$$

$$2)\ case\ iff\ x=2.\ Then:$$

$$\frac{1}{2+y}+\frac{1}{2+y+z}+\frac{1}{2+y+z+t}=\frac{1}{2}$$

$$\frac{3}{2+y}>\frac{1}{2}\rightarrow\ y+2<6\rightarrow\boxed{y<4}$$

$$iff\ x=2;\ y=1.\ Then:$$

$$\frac{1}{3+z}+\frac{1}{3+z+t}=\frac{1}{6}\rightarrow\frac{2}{3+z}>\frac{1}{6}\rightarrow\boxed{z<9}$$

$$The\ set\ of\ solutions\ satisfying\ the\ condition:$$

$$x=2;\ y=1,\quad \boxed{z<9}\rightarrow\{(2;1;4;35),(2;1;5;16),(2;1;6;9),(2;1;7;5)\}$$

$$3)\ case.\ For\ the\ condition\ x=2.\ Let's\ look\ at\ the\ case\ y=2$$

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{4+z}+\frac{1}{4+z+t}=1\rightarrow\frac{1}{4+z}+\frac{1}{4+z+t}=\frac{1}{4}\rightarrow\frac{2}{4+z}>\frac{1}{4}\rightarrow\boxed{z<4}$$

$$So,\quad x=2,\quad y=2\ and\ z<4$$

$$In\ this\ case\ ,\quad a\ set\ of\ solutions:\ \{(2;2;1;15),(2;2;2;6)\}$$

$$4)\ case.\ \ x=2\ ,\quad y=3$$

$$\frac{1}{2}+\frac{1}{5}+\frac{1}{7+z}+\frac{1}{7+z+t}=1,\qquad \frac{1}{7+z}+\frac{1}{7+z+t}=\frac{10}{3}\rightarrow\frac{2}{7+z}>\frac{3}{10}\rightarrow z<-\frac{1}{3}$$

$$So, in\ this\ case\ there\ is\ no\ solution.$$

$$5)\ case.\ x=3$$

$$\frac{1}{3}+\frac{1}{3+y}+\frac{1}{3+y+z}+\frac{1}{3+y+z+t}=1\rightarrow\frac{1}{3+y}+\frac{1}{3+y+z}+\frac{1}{3+y+z+t}=\frac{2}{3}$$

$$\rightarrow\frac{3}{3+y}>\frac{2}{3}\rightarrow 6+2y<9\rightarrow\boxed{y<1.5}$$

$$iff\ \ y=1.\ Then\ \frac{1}{3}+\frac{1}{4}+\frac{1}{4+z}+\frac{1}{4+z+t}=1\rightarrow\frac{1}{4+z}+\frac{1}{4+z+t}=\frac{5}{12}$$

$$\rightarrow\frac{2}{4+z}>\frac{5}{12}\rightarrow 20+5z<24\rightarrow 5z<4\rightarrow\boxed{z<\frac{4}{5}}$$

$$The\ resulting\ answer:$$

$$\{(2;1;4;35),(2;1;5;16),(2;1;6;9),(2;1;7;5),(2;2;1;15),(2;2;2;6)\}$$