ROMANIAN MATHEMATICAL MAGAZINE

Find all values of $x, y, z \in \mathbb{Z}$ such that :

$$x^4 + 9y^2 + 25z^2 = x^2 + 6xy + 2022$$

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The equation is equivalent to :

$$9y^2 - 6xy + x^4 - x^2 - 2022 + 25z^2 = 0.$$

This equation has integral solutions if and only if its discriminant

$$36[2023 - (x^2 - 1)^2 - 25z^2]$$

is a perfect square. It follows that

$$2023 - (x^2 - 1)^2 - 25z^2 = t^2 \text{ or } (x^2 - 1)^2 + 25z^2 + t^2 = 2023.$$

For any integer n, we have $n^2 \equiv 0, 1, 4 \ (mod 8)$, then

$$(x^2 - 1)^2 + 25z^2 + t^2 \equiv 0, 1, 2, 3, 4, 5, 6 \ (mod 8),$$

but we have $2023 \equiv 7 \ (mod 8)$. So the equation has no solution in integers.