ROMANIAN MATHEMATICAL MAGAZINE

Solve for real numbers:

$$4^{\cos^2\left(\frac{x}{2}\right)}+\cos x=4$$

Proposed by Khaled Abd Imouti-Syria

Solution by Pham Duc Nam-Vietnam

$$4^{\cos^2\left(\frac{x}{2}\right)}+\cos(x)=4\Leftrightarrow 2^{1+\cos(x)}+\cos(x)=4, t=\cos(x), -1\le t\le 1$$

$\Leftrightarrow 2^{1+t}+t=4, f(t)=2^{1+t}+t, f'(t)=2^{t+1}\ln(2)+1>0, \forall t\in[-1,1]\Rightarrow f(t)$ is strictly increasing on $[-1,1]\Rightarrow 2^{1+t}+t=4$ has unique root, and $g(t)=2^{1+t}+t-4, g(0)=-2, g(1)=1\Rightarrow$ The root lies on $(0,1)$.

We have: $2^{1+t}+t=4\Leftrightarrow 2^{1+t}=4-t\Leftrightarrow \ln(2)\,2^{1+t}=\ln(2)\,(4-t)\Leftrightarrow$

$$\Leftrightarrow 2^{4-t}\ln(2)\,(4-t)=32\ln(2)\Leftrightarrow$$

$$\Leftrightarrow \ln(2)\,(4-t)e^{(\ln(2)(4-t))}=32\ln(2)\overset{xe^x=z\Rightarrow x=W(z)}{\rightarrow} W\left(\ln(2)(4-t)e^{(\ln(2)(4-t))}\right)=$$

$$=W(32\ln(2))$$

$$\Leftrightarrow \ln(2)\,(4-t)=W(32\ln(2))\Leftrightarrow t=4-\frac{W(32\ln(2))}{\ln(2)}\approx 0.7156207332755864$$

$\Rightarrow x=\pm\arccos\left(4-\frac{W(32\ln(2))}{\ln(2)}\right)+k2\pi, k\in\mathbb{Z}. W(z)$ is the Lambert W function.