ROMANIAN MATHEMATICAL MAGAZINE

Find all values of $x, y, z \in \mathbb{R}$ such that :

$$xyz(xy+yz+zx)=x^2+y^2+z^2=3$$

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$$a^2+b^2+c^2-ab-bc-ca=\frac{1}{2}((a-b)^2+(b-c)^2+(c-a)^2)\geq 0$$
$$\Rightarrow a^2+b^2+c^2\geq ab+bc+ca\ \forall\, a,b,c\in\mathbb{R}\rightarrow(1)$$
$$\text{Putting } a=xy, b=yz, c=zx \text{ in } (1), \sum_{cyc}x^2y^2\geq xyz\sum_{cyc}x$$
$$\forall\, x,y,z\in\mathbb{R}\rightarrow\ (2)\ ('' = '' \text{ iff } x=y=z) \text{ and}$$
$$\text{putting } a=x^2, b=y^2, c=z^2 \text{ in } (1), \sum_{cyc}x^4\geq\sum_{cyc}x^2y^2\overset{\text{via }(2)}{\geq}xyz\sum_{cyc}x$$
$$\forall\, x,y,z\in\mathbb{R}\rightarrow(3)\ ('' = '' \text{ iff } x=y=z)$$
$$\therefore (2)+(3)\Rightarrow\sum_{cyc}x^4+\sum_{cyc}x^2y^2\geq 2xyz\sum_{cyc}x\Rightarrow\sum_{cyc}x^4+2\sum_{cyc}x^2y^2\geq$$
$$\sum_{cyc}x^2y^2+2xyz\sum_{cyc}x\Rightarrow\left(\sum_{cyc}x^2\right)^2\boxed{\geq}\left(\sum_{cyc}xy\right)^2$$
$$\forall\, x,y,z\in\mathbb{R}\ \rightarrow(4)\ ('' = '' \text{ iff } x=y=z)$$
$$\text{Now}, \forall\, a,b,c\geq 0, \sum_{cyc}a\geq 3.\sqrt[3]{abc}\ \therefore \text{ assigning } a=x^2, b=y^2, c=z^2,$$
$$\sum_{cyc}x^2\geq 3.\sqrt[3]{x^2y^2z^2}\Rightarrow\left(\sum_{cyc}x^2\right)^3\boxed{\geq}27x^2y^2z^2$$
$$\forall\, x,y,z\in\mathbb{R}\ \rightarrow(5)\ ('' = '' \text{ iff } x^2=y^2=z^2)$$
$$\therefore (2)\bullet(3)\Rightarrow\left(\sum_{cyc}x^2\right)^5\geq 27x^2y^2z^2\left(\sum_{cyc}xy\right)^2\ \forall\, x,y,z\in\mathbb{R}$$
$$('' = '' \text{ iff } x=y=z)\overset{x^2+y^2+z^2=3}{\Rightarrow}27\left(\sum_{cyc}x^2\right)^2\geq 27x^2y^2z^2\left(\sum_{cyc}xy\right)^2$$
$$\Rightarrow\boxed{\left(\sum_{cyc}x^2\right)^2\geq x^2y^2z^2\left(\sum_{cyc}xy\right)^2\ \forall\, x,y,z\in\mathbb{R}\ ('' = '' \text{ iff } x=y=z)}\ \text{but,}$$

$$\left(\sum_{cyc} x^2\right)^2 = x^2y^2z^2\left(\sum_{cyc} xy\right)^2 \ (\because xyz(xy+yz+zx) = x^2+y^2+z^2)$$

$$\therefore xyz\left(\sum_{cyc} xy\right) = \sum_{cyc} x^2 = 3 \Rightarrow x = y = z \therefore x^3.3x^2 = 3$$

$$\left(\text{using } xyz\left(\sum_{cyc} xy\right) = 3\right) \Rightarrow x = 1$$

$\therefore$ $(x = y = z = 1)$ **is the only desired set of values** (ans)