ROMANIAN MATHEMATICAL MAGAZINE

Find *all* values of $x, y, z \in \mathbb{R}$ such that :

$$x^3y^3z^3(x^2y^2 + y^2z^2 + z^2x^2) = x^2 + y^2 + z^2 = 3$$

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$$\forall\, a, b, c \ge 0, \sum_{cyc} a \ge 3.\sqrt[3]{abc} \therefore \text{assigning } a = x^2, b = y^2, c = z^2,$$

$$3 = \sum_{cyc} x^2 \ge 3.\sqrt[3]{x^2y^2z^2} \Rightarrow x^2y^2z^2 \le 1 \Rightarrow xyz \le 1 \to (1)$$

$$\left(\because x^3y^3z^3\left(\sum_{cyc} x^2y^2\right) = 3 \Rightarrow x^3y^3z^3 > 0 \Rightarrow xyz > 0\right)$$

$$\text{Now, } \sum_{cyc} x^2 = x^3y^3z^3\left(\sum_{cyc} x^2y^2\right) \overset{\text{via } (2)}{\le} \sum_{cyc} x^2y^2 \overset{x^2+y^2+z^2=3}{\Rightarrow}$$

$$\left(\sum_{cyc} x^2\right)^2 \le 3\sum_{cyc} x^2y^2 \Rightarrow \sum_{cyc} x^4 - \sum_{cyc} x^2y^2 \le 0 \Rightarrow \boxed{\frac{1}{2}\sum_{cyc}(x^2 - y^2)^2 \le 0};$$

$$\text{but } \frac{1}{2}\sum_{cyc}(x^2 - y^2)^2 \ge 0 \therefore \sum_{cyc}(x^2 - y^2)^2 = 0 \Rightarrow x^2 = y^2 = z^2$$

$$\Rightarrow 3x^2 = 3 \left(\text{using } \sum_{cyc} x^2 = 3\right) \therefore \boxed{x = \pm 1, y = \pm 1, z = \pm 1} \to (2)$$

$$\text{Using } x^3y^3z^3\left(\sum_{cyc} x^2y^2\right) = 3, \text{we get : } 3xyz = 3\ (\because x^2 = y^2 = z^2 = 1)$$

$$\Rightarrow \boxed{xyz = 1} \to (3) \therefore (2)\ and\ (3) \Rightarrow$$

$$\begin{pmatrix} x = 1 \\ y = 1 \\ z = 1 \end{pmatrix}, \begin{pmatrix} x = 1 \\ y = -1 \\ z = -1 \end{pmatrix}, \begin{pmatrix} x = -1 \\ y = 1 \\ z = -1 \end{pmatrix} \text{ and } \begin{pmatrix} x = -1 \\ y = -1 \\ z = 1 \end{pmatrix}$$

$$\text{are the only set of values} \mid x^3y^3z^3(x^2y^2 + y^2z^2 + z^2x^2) = x^2 + y^2 + z^2 = 3\ (ans)$$