ROMANIAN MATHEMATICAL MAGAZINE

Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$(9x)+f(16x)=2f(12x)\qquad,\quad \forall\in\mathbb{R}$$

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$$f(9x)+f(16x)=2f(12x)\qquad \forall\in\mathbb{R}$$

$$f\left(\frac{5}{12}\right)+f\left(\frac{16}{12}\right)=2f(x)$$

$$\boxed{2f(x)=f\left(\frac{3}{4}x\right)+f\left(\frac{4}{3}x\right)}$$

$$f(x)=\frac{1}{2}\left[f\left(\frac{3}{4}x\right)+f\left(\frac{4}{3}x\right)\right]$$

$$f\left(\frac{4}{3}x\right)=\frac{1}{2}\left[f(x)+f\left(\frac{16}{9}x\right)\right]$$

$$\boxed{f(x)=2f\left(\frac{4}{3}x\right)-f\left(\frac{16}{9}x\right)}$$

In similar way:

$$\boxed{f(x)=2f\left(\frac{3}{4}x\right)-f\left(\frac{9}{16}x\right)}$$

$$2f(x)=2\left[f\left(\frac{4}{3}x\right)+f\left(\frac{3}{4}x\right)\right]-\left[f\left(\frac{16}{9}x\right)+f\left(\frac{9}{16}x\right)\right]$$

$$2f(x)=2\big(2f(x)\big)-\left(f\left(\frac{16}{9}x\right)+f\left(\frac{9}{16}x\right)\right)$$

$$\boxed{2f(x)=f\left(\frac{16}{9}x\right)+f\left(\frac{9}{16}x\right)}$$

So from (*) and (**) :

$$2f(x)=f\left(\left(\frac{4}{3}\right)^{n}x\right)+f\left(\left(\frac{3}{4}\right)^{n}x\right)\quad,\ \forall\, n\geq 1$$

$$2f\left(\left(\frac{3}{4}\right)^{n}x\right)=f(x)+f\left(\left(\frac{3}{4}\right)^{2n}x\right)\qquad(***)$$

because f is continuous by taking limits of two sides:

$$from\ (***):2f(0)=f(x)+f(0)$$

so : $f(x)=f(0)=c$

so : f is constant