ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC holds:

$$\sum_{cyc} a^3 = 6sR^2 \Rightarrow \prod_{cyc}(b^2+c^2-bc-a^2) = 0$$

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$$(b^2+c^2-bc-a^2) = b^2+c^2-a^2-bc = 2bc\cos A - bc = bc(2\cos A - 1)$$

(analog)

$$\therefore \prod_{cyc} bc(2\cos A - 1)$$

$$= ab\cdot bc\cdot ca(2\cos A-1)(2\cos B-1)(2\cos C-1)$$

$$= a^2b^2c^2\{-(1-2\cos A)(1-2\cos B)(1-2\cos C)\}$$

$$= -a^2b^2c^2\begin{bmatrix} 1-2(\cos A+\cos B+\cos C)+ \\ +4(\cos A\cos B+\cos B\cos C+\cos C\cos A)- \\ -8\cos A\cos B\cos C \end{bmatrix}$$

$$= -a^2b^2c^2\left[1-2\left(1+\frac{r}{R}\right)+4\frac{s^2+r^2-4R^2}{4R^2}-8\frac{s^2-(2R+r)^2}{4R^2}\right]$$

$$= -\frac{a^2b^2c^2}{R^2}[R^2-2R^2-2Rr+s^2+r^2-4R^2-2s^2+2(2R+r)^2]$$

$$= -\frac{a^2b^2c^2}{R^2}[-R^2-2Rr+s^2+r^2-4R^2-2s^2+8R^2+8Rr+2r^2]$$

$$= -\frac{a^2b^2c^2}{R^2}[3R^2+6Rr-s^2+3r^2] = 0$$

Note: $\sum a^3 = 6sR^2,\ \therefore 2s(s^2-3r^2-6Rr) = 6sR^2,\ \therefore 3R^2+6Rr+3r^2-s^2 = 0$

Using the following relationships:

$$\cos A+\cos B+\cos C = 1+\frac{r}{R},\qquad \sum \cos A\cos B = \frac{s^2+r^2-4R^2}{4R^2}$$

$$\prod \cos A = \frac{s^2-(2R+r)^2}{4R^2}$$