ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC holds:

$$\sqrt{2}a\cos\frac{B}{2}\cos\frac{C}{2}=s\Rightarrow \sec(2B)+\tan(2B)=\frac{c+b}{c-b}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\sqrt{2}a\sqrt{\frac{s(s-b)}{ab}}\cdot\sqrt{\frac{s(s-c)}{ac}}=s\Rightarrow\sqrt{2}\sqrt{\frac{(s-b)(s-c)}{bc}}=1$$

$$\Rightarrow \sin\frac{A}{2}=1\Rightarrow\frac{A}{2}=\frac{\pi}{4}\Rightarrow A=\frac{\pi}{2}$$

Now, $\sec(2B)+\tan(2B)=\frac{1+\sin(2B)}{\cos(2B)}=\frac{(\sin B+\cos B)^2}{\cos^2 B-\sin^2 B}=\frac{\cos B+\sin B}{\cos B-\sin B}=\frac{\sin C+\sin B}{\sin C-\sin B}$

$$\left[\because B+C=\frac{\pi}{2}\right]$$

$=\frac{c+b}{c-b}$ [using law of sines]

Solution 2 by Cosqun Memmedoff-Azerbaijan

$$\sqrt{2}\,a\cos\frac{B}{2}\cos\frac{C}{2}=s,\ s=\frac{a+b+c}{2}$$

$$s^2=2a^2\cos^2\frac{B}{2}\cos^2\frac{C}{2}$$

$$s^2=2a^2\left(\frac{1+\cos B}{2}\right)\left(\frac{1+\cos C}{2}\right),\ \cos B=\frac{a^2+c^2-b^2}{2ac},\ \cos C=\frac{a^2+b^2-c^2}{2ab}$$

$$s^2=2a^2\left(\frac{1}{2}+\frac{a^2+c^2-b^2}{4ac}\right)\left(\frac{1}{2}+\frac{a^2+b^2-c^2}{4ab}\right)$$

$$s^2=2a^2\left(\frac{(a+c)^2-b^2}{4ac}\right)\left(\frac{(a+b)^2-c^2}{4ab}\right)$$

$$s^2=\frac{(a+b+c)^2(a+c-b)(a+b-c)}{8bc}$$

$$\frac{(a+b+c)^2}{4}=\frac{(a+b+c)^2(a+c-b)(a+b-c)}{8bc}$$

$$(a+c-b)(a+b-c)=2bc$$

$$a^2+ab-ac+ca+cb-c^2-ab-b^2+bc=2bc$$

$$a^2-c^2-b^2=0\Leftrightarrow a^2=b^2+c^2\Rightarrow\cos B=\frac{c}{a},\ \sin B=\frac{b}{a}$$

$$\sec(2B)+\tan(2B)=\frac{1}{\cos(2B)}+\frac{\sin(2B)}{\cos(2B)}=\frac{\sin(2B)+1}{\cos 2B}=$$

$$=\frac{\cos B+\sin B}{\cos B-\sin B}=\frac{\frac{c}{a}+\frac{b}{a}}{\frac{c}{a}-\frac{b}{a}}=\frac{c+b}{c-b}$$

Solution 3 by Tapas Das-India

$$\sqrt{2}a\cos\frac{B}{2}\cos\frac{C}{2}=s$$

$$\sqrt{2}\cdot 2R\cdot\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)\cos\frac{B}{2}\cdot\cos\frac{C}{2}=s$$

$$2\sqrt{2}\times 2\times R\cdot\sin\frac{A}{2}\cdot\frac{s}{4R}=s$$

$$\sqrt{2}\sin\frac{A}{2}=1\therefore\sin\frac{A}{2}=\frac{1}{\sqrt{2}}=\sin\frac{\pi}{4}$$

$$A=\frac{\pi}{2}$$

$$\therefore\cos B=\frac{c}{a}$$

$$\sin B=\frac{b}{a}$$

$$\sec 2B+\tan 2B=\frac{1+\sin 2B}{\cos 2B}=\frac{(\sin B+\cos B)^2}{(\cos B+\sin B)(\cos B-\sin B)}$$

$$=\frac{\cos B+\sin B}{\cos B-\sin B}=\frac{\frac{c}{a}+\frac{b}{a}}{\frac{c}{a}-\frac{b}{a}}=\frac{c+b}{c-b}$$