ROMANIAN MATHEMATICAL MAGAZINE

In any scalene Δ ABC, the following relationship holds :

$$\frac{b+c}{(r_a-r_b)(r_a-r_c)}+\frac{a+c}{(r_b-r_a)(r_b-r_c)}+\frac{a+b}{(r_c-r_a)(r_c-r_b)}=\frac{2}{a+b+c}$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution 1 by proposer, Solution 2 by Soumava Chakraborty-Kolkata-India

Solution 1 by proposer

$$\sum\frac{b+c}{(r_a-r_b)(r_a-r_c)}=\sum\frac{2p-a}{p^2r^2\left(\frac{1}{p-a}-\frac{1}{p-b}\right)\left(\frac{1}{p-a}-\frac{1}{p-c}\right)}$$

$$=\frac{1}{p^2r^2}\cdot\sum\frac{(2p-a)(p-a)^2(p-b)(p-c)}{(a-b)(a-c)}=\frac{pr^2}{p^2r^2}\cdot\sum\frac{(2p-a)(p-a)}{(a-b)(a-c)}$$

$$=\frac{1}{p}\sum\frac{2p^2-3pa+a^2}{(a-b)(a-c)}=$$

$$=\frac{1}{p}\left\{2p^2\cdot\sum\underbrace{\frac{1}{(a-b)(a-c)}}_{0}-3p\cdot\sum\underbrace{\frac{a}{(a-b)(a-c)}}_{0}+\sum\underbrace{\frac{a^2}{(a-b)(a-c)}}_{1}\right\}$$

$$=\frac{1}{p}\sum\frac{a^2}{(a-b)(a-c)}=\frac{1}{p}\cdot 1=\frac{2}{2p}=\frac{2}{a+b+c}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{b+c}{(r_a-r_b)(r_a-r_c)}=\frac{b+c}{\left(\frac{rs}{s-a}-\frac{rs}{s-b}\right)\left(\frac{rs}{s-a}-\frac{rs}{s-c}\right)}$$

$$=\frac{(b+c)(s-a)^2(s-b)(s-c)}{r^2s^2(a-b)(a-c)}=\frac{r^2s(b+c)(s-a)(c-b)}{r^2s^2(a-b)(b-c)(c-a)}$$

$$=\frac{(b+c)(s-a)\big((s-b)-(s-c)\big)}{s(a-b)(b-c)(c-a)}$$

$$=\frac{1}{s(a-b)(b-c)(c-a)}\cdot\begin{pmatrix}\big(s+(s-a)\big)(s-a)(s-b)\\-\big(s+(s-a)\big)(s-a)(s-c)\end{pmatrix}$$

$$=\frac{s(s-a)(s-b)+(s-a)^2(s-b)-s(s-a)(s-c)-(s-a)^2(s-c)}{s\big((s-b)-(s-a)\big)\big((s-c)-(s-b)\big)\big((s-a)-(s-c)\big)}$$

$$=\frac{s\big((s-a)(s-b)-(s-a)(s-c)\big)+x^2y-x^2z}{s(y-x)(z-y)(x-z)}\text{ and analogs}$$

$$(x=s-a,y=s-b,z=s-c)$$

$$\therefore \frac{b+c}{(r_a-r_b)(r_a-r_c)}+\frac{a+c}{(r_b-r_a)(r_b-r_c)}+\frac{a+b}{(r_c-r_a)(r_c-r_b)}$$

$$=\frac{s\sum_{cyc}(s-a)(s-b)-s\sum_{cyc}(s-a)(s-c)+x^2y+y^2z+z^2x-xy^2-yz^2-zx^2}{s(y-x)(z-y)(x-z)}$$

$$=\frac{xy(x-y)-z(x-y)(x+y)+z^2(x-y)}{s(x-y)(y-z)(x-z)}=\frac{(x-y)\left(xy-zx-zy+z^2\right)}{s(x-y)(y-z)(x-z)}$$

$$=\frac{(x-y)\big(x(y-z)-z(y-z)\big)}{s(x-y)(y-z)(x-z)}=\frac{(x-y)(y-z)(x-z)}{s(x-y)(y-z)(x-z)}=\frac{2}{2s}=\frac{2}{a+b+c}\ \text{(QED)}$$