ROMANIAN MATHEMATICAL MAGAZINE

In any scalene $\Delta$ ABC, the following relationship holds :

$$\frac{(b+c)^2}{(h_a-h_b)(h_c-h_a)}+\frac{(a+c)^2}{(h_b-h_a)(h_c-h_b)}+\frac{(a+b)^2}{(h_c-h_a)(h_b-h_c)}=\frac{2R}{r}$$

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$$\frac{(b+c)^2}{(h_a-h_b)(h_c-h_a)}+\frac{(a+c)^2}{(h_b-h_a)(h_c-h_b)}+\frac{(a+b)^2}{(h_c-h_a)(h_b-h_c)}$$

$$=\frac{(b+c)^2}{\left(\frac{2rs}{a}-\frac{2rs}{b}\right)\left(\frac{2rs}{c}-\frac{2rs}{a}\right)}+\frac{(a+c)^2}{\left(\frac{2rs}{b}-\frac{2rs}{a}\right)\left(\frac{2rs}{c}-\frac{2rs}{b}\right)}+\frac{(a+b)^2}{\left(\frac{2rs}{c}-\frac{2rs}{a}\right)\left(\frac{2rs}{b}-\frac{2rs}{c}\right)}$$

$$=\frac{a(b+c)^2}{\frac{4r^2s^2}{4Rrs}(a-b)(c-a)}+\frac{b(a+c)^2}{\frac{4r^2s^2}{4Rrs}(a-b)(b-c)}+\frac{c(a+b)^2}{\frac{4r^2s^2}{4Rrs}(b-c)(c-a)}=$$

$$\frac{R}{rs(a-b)(b-c)(c-a)}\left(a(b-c)(b+c)^2+b(c-a)(a+c)^2+c(a-b)(a+b)^2\right)$$

$$=\frac{R\left(a(b+c)(b^2-c^2)+b(a+c)(c^2-a^2)+c(a+b)(a^2-b^2)\right)}{rs(a-b)(b-c)(c-a)}$$

$$=\frac{R\left(\left(\sum_{cyc}ab-bc\right)(b^2-c^2)+\left(\sum_{cyc}ab-ca\right)(c^2-a^2)+\left(\sum_{cyc}ab-ab\right)(a^2-b^2)\right)}{rs(a-b)(b-c)(c-a)}$$

$$=\frac{R\left(\left(\sum_{cyc}ab\right)(b^2-c^2+c^2-a^2+a^2-b^2)-\left(bc(b^2-c^2)+c^3a-ca^3+a^3b-ab^3\right)\right)}{rs(a-b)(b-c)(c-a)}$$

$$=\frac{-R}{rs(a-b)(b-c)(c-a)}\left(bc(b^2-c^2)-a(b-c)(b^2+bc+c^2)+a^3(b-c)\right)$$

$$=\frac{-R(b-c)}{rs(a-b)(b-c)(c-a)}\left(b^2c+bc^2-ab^2-abc-ac^2+a^3\right)$$

$$=\frac{-R(b-c)}{rs(a-b)(b-c)(c-a)}\left(b^2(c-a)-a(c-a)(c+a)+bc(c-a)\right)$$

$$=\frac{-R(b-c)(c-a)}{rs(a-b)(b-c)(c-a)}\left(b^2-ca-a^2+bc\right)$$

$$=\frac{-R(b-c)(c-a)}{rs(a-b)(b-c)(c-a)}\left((b+a)(b-a)+c(b-a)\right)$$

$$=\frac{-R(b-c)(c-a)(b-a)(a+b+c)}{rs(a-b)(b-c)(c-a)}=\frac{R(2s)}{rs}=\frac{2R}{r}\ \text{(QED)}$$