ROMANIAN MATHEMATICAL MAGAZINE

Let ΔDEF be the orthic triangle of acute $\Delta ABC, D \in (BC), E \in (CA)$, $F \in (AB), r_1, r_2, r_3$ –inradii of $\Delta AFE, \Delta BDF, \Delta CED$ respectively. Prove that:

$$\frac{a}{r_1} + \frac{b}{r_2} + \frac{c}{r_3} = \frac{R}{r} \cdot \sum_{cyc} \tan A$$

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Solution by Daniel Sitaru-Romania

$$AF = AC\cos A = b\cos A, \qquad AE = AB\cos A = c\cos A$$

$$EF^2 = AF^2 + AE^2 - 2AF \cdot AE\cos A = b^2\cos^2 A + c^2\cos^2 A - 2bc\cos^3 A =$$

$$= (b^2 + c^2 - 2bc\cos A)\cos^2 A = a^2\cos^2 A$$

$$EF = a\cos A$$

$$r_1 = \frac{[AEF]}{\frac{AE+AF+EF}{2}} = \frac{\frac{1}{2} \cdot AE \cdot AF \cdot \sin A}{\frac{a\cos A + b\cos A + c\cos A}{2}} = \frac{b\cos A \cdot c\cos A \cdot \sin A}{(a+b+c)\cos A} =$$

$$= \frac{bc\sin A \cdot \cos A}{2s} = \frac{4F\cos A}{2s} = \frac{2rs\cos A}{s} = 2r\cos A$$

$$\frac{a}{r_1} + \frac{b}{r_2} + \frac{c}{r_3} = \frac{2R\sin A}{2r\cos A} + \frac{2R\sin B}{2r\cos B} + \frac{2R\sin C}{2r\cos C} = \frac{R}{r} \cdot \sum_{cyc} \tan A$$