ROMANIAN MATHEMATICAL MAGAZINE

In any bicentric quadrilateral ABCD with sides a, b, c, d, the following relationship holds:

$$3R\sqrt{2}.\min\left(\frac{1}{a^{-3}+b^{-3}+c^{-3}},\frac{1}{a^{-3}+b^{-3}+d^{-3}},\frac{1}{a^{-3}+c^{-3}+d^{-3}},\frac{1}{b^{-3}+c^{-3}+d^{-3}}\right)$$
$$\le F^2 \le \frac{R\sqrt{2}}{3}.\max(a^3+b^3+c^3, a^3+b^3+d^3, a^3+c^3+d^3, b^3+c^3+d^3)$$

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Via Brahmagupta and Parameshvara,

$$16F^2R^2 = (ac+bd)(ab+cd)(ad+bc)$$
$$\Rightarrow x\left(bd\left((a+c)^2-2ac\right)+ac\left((b+d)^2-2bd\right)\right) = 16R^2r^2s^2 \ (x = ac+bd)$$
$$\Rightarrow x\left(bd(s^2-2ac)+ac(s^2-2bd)\right) = 16R^2r^2s^2 \Rightarrow x(s^2x-4r^2s^2) = 16R^2r^2s^2$$
$$\Rightarrow x^2-4x.r^2-16R^2r^2 = 0 \Rightarrow x = \frac{4r^2\pm\sqrt{64R^2r^2+16r^4}}{2}$$
$$\Rightarrow ac+bd = 2r^2+2r*\sqrt{4R^2+r^2} \to (1)$$

Now, $(bs+ca)(as+bd)(ds+ca)(cs+bd)$

$$= s^3(a^2bc^2+a^2c^2d+ab^2d^2+b^2cd^2)$$
$$+s^2(a^3c^3+b^3d^3+a^2b^2cd+a^2bcd^2+ab^2c^2d+abc^2d^2)$$
$$+sabcd(a^2c+ac^2+b^2d+bd^2)+abcdr^2s^2+abcds^4$$
$$\ge s^4(a^2c^2+b^2d^2)+s^2abcd(ac+bd)+s^2abcd(as+cs)+sabcd(acs+bds)$$
$$+abcdr^2s^2+abcds^4 \overset{A-G}{\ge} 2s^4abcd+2s^2abcd(ac+bd)+2abcds^4+abcdr^2s^2$$
$$= s^2abcd(4s^2+2(ac+bd)+r^2) \overset{\text{via }(1)}{=}$$
$$r^2s^4\left(4s^2+2\left(2r^2+2r*\sqrt{4R^2+r^2}\right)+r^2\right) \Rightarrow$$
$$(bs+ca)(as+bd)(ds+ca)(cs+bd) \ge r^2s^4\left(4s^2+4r*\sqrt{4R^2+r^2}+5r^2\right) \to (2)$$

We have : $m^4 =$

$$\left(\min\left(\frac{1}{a^{-3}+b^{-3}+c^{-3}},\frac{1}{a^{-3}+b^{-3}+d^{-3}},\frac{1}{a^{-3}+c^{-3}+d^{-3}},\frac{1}{b^{-3}+c^{-3}+d^{-3}}\right)\right)^4$$
$$\le \frac{(a^3b^3c^3)(a^3b^3d^3)(a^3c^3d^3)(b^3c^3d^3)}{(a^3b^3+b^3c^3+c^3a^3)(a^3b^3+b^3d^3+d^3a^3)(a^3c^3+c^3d^3+d^3a^3)(b^3c^3+c^3d^3+b^3d^3)}$$
$$\le \frac{9^4.F^{18}}{(bs+ca)^3(as+bd)^3(ds+ca)^3(cs+bd)^3}$$

$$\left(\begin{array}{c} \because a^3b^3+b^3c^3+c^3a^3 = c^3a^3+b^3(c+a)(c^2-ca+a^2) \\ \ge c^3a^3+\dfrac{b^3}{4}(c+a)^3 = c^3a^3+\dfrac{b^3s^3}{8}+\dfrac{b^3s^3}{8} \overset{\text{Holder}}{\ge} \dfrac{\left(\frac{bs}{2}+\frac{bs}{2}+ca\right)^3}{9} \\ \Rightarrow a^3b^3+b^3c^3+c^3a^3 \ge \dfrac{(bs+ca)^3}{9} \text{analogs} \end{array}\right)$$

$$\leq \frac{9^4.F^{18}}{r^6s^{12}\left(4s^2+4r*\sqrt{4R^2+r^2}+5r^2\right)^3} \overset{?}{\leq} \left(\frac{F^2}{3R\sqrt{2}}\right)^4 = \frac{F^8}{81R^4.4}$$

$$\Leftrightarrow \frac{9^6.r^{10}s^{10}}{r^6s^{12}\left(4s^2+4r*\sqrt{4R^2+r^2}+5r^2\right)^3} \overset{?}{\leq} \frac{1}{4R^4}$$

$$\Leftrightarrow s^2\left(4s^2+4r*\sqrt{4R^2+r^2}+5r^2\right)^3 \underset{(*)}{\overset{?}{\geq}} 9^6.4R^4r^4$$

Now, via Blundon – Eddy, LHS of $(*) \geq$

$$8r\left(\sqrt{4R^2+r^2}-r\right)\left(4.8r\left(\sqrt{4R^2+r^2}-r\right)+4r*\sqrt{4R^2+r^2}+5r^2\right)^3 \overset{?}{\geq} 9^6.4R^4r^4$$

$$\Leftrightarrow 2(4R^2+r^2)(256R^2+172r^2)+2r\left(192R^2r+75r^3+96r(4R^2+r^2)\right)-729R^4$$

$$\overset{?}{\geq} 2r.\sqrt{4R^2+r^2}.\left(256R^2+172r^2+192R^2+75r^2+96(4R^2+r^2)\right)$$

$$\Leftrightarrow (1319R^4+3040R^2r^2+686r^4)^2 \overset{?}{\geq} \left(2r.\sqrt{4R^2+r^2}.(832R^2+343r^2)\right)^2$$

$$\Leftrightarrow 1739761t^3-3056064t^2-849660t+5488 \overset{?}{\geq} 0 \quad \left(t=\frac{R^2}{r^2}\right)$$

$$\Leftrightarrow (t-2)\left(1739761t^2+422086t+1372(t-2)\right) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{L.Fejes Toth}}{\geq} 2$$

$$\Rightarrow (*) \text{ is true} \Rightarrow m^4 \leq \left(\frac{F^2}{3R\sqrt{2}}\right)^4$$

$$\therefore \boxed{3R\sqrt{2}.\min\left(\frac{1}{a^{-3}+b^{-3}+c^{-3}},\frac{1}{a^{-3}+b^{-3}+d^{-3}},\frac{1}{a^{-3}+c^{-3}+d^{-3}},\frac{1}{b^{-3}+c^{-3}+d^{-3}}\right) \leq F^2}$$

Also, $\frac{R\sqrt{2}}{3}.max(a^3+b^3+c^3,a^3+b^3+d^3,a^3+c^3+d^3,b^3+c^3+d^3) \geq$

$$\frac{R\sqrt{2}}{4.3}(a^3+b^3+c^3+a^3+b^3+d^3+a^3+c^3+d^3+b^3+c^3+d^3) \overset{A-G}{\geq}$$

$$\frac{R\sqrt{2}}{4}(abc+abd+acd+bcd) = \frac{R\sqrt{2}}{4}(acs+bds) \overset{A-G}{\geq} \frac{Rs\sqrt{2}}{4}.2\sqrt{acbd} =$$

$$\frac{Rrs^2}{\sqrt{2}} \overset{?}{\geq} F^2 = r^2s^2 \Leftrightarrow R \overset{?}{\geq} \sqrt{2}r \rightarrow \text{true via L. Fejes Toth}$$

$$\therefore \boxed{F^2 \leq \frac{R\sqrt{2}}{3}.max(a^3+b^3+c^3,a^3+b^3+d^3,a^3+c^3+d^3,b^3+c^3+d^3)} \text{ (QED)}$$