ROMANIAN MATHEMATICAL MAGAZINE

In any bicentric quadrilateral $ABCD$ with $AB = a, BC = b, CD = c, DA = d, AC = e,$ $BD = f, m = \min(e,f)$ and $\frac{R}{r} = \alpha \ge \sqrt{2}$, the following relationship holds :

$$a^2 + b^2 + c^2 + d^2 \ge 4\sqrt[3]{\frac{\alpha^2}{m^2}}.F^{\frac{4}{3}}$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

Via Brahmagupta and Parameshvara, $16F^2R^2$

$$= (ac+bd)(ab+cd)(ad+bc) \Rightarrow x\left(bd\left((a+c)^2 - 2ac\right) + ac\left((b+d)^2 - 2bd\right)\right)$$

$$= 16R^2r^2s^2 \ (x = ac+bd) \Rightarrow x\left(bd\left(s^2 - 2ac\right) + ac\left(s^2 - 2bd\right)\right) = 16R^2r^2s^2$$

$$\Rightarrow x\left(s^2x - 4r^2s^2\right) = 16R^2r^2s^2 \Rightarrow x^2 - 4x.r^2 - 16R^2r^2 = 0$$

$$\Rightarrow x = \frac{4r^2 \pm \sqrt{64R^2r^2 + 16r^4}}{2} \Rightarrow ac + bd = 2r^2 + 2r * \sqrt{4R^2 + r^2} \to (1)$$

$$\therefore \sum_{cyc} a^2 = 4s^2 - 2\left(ac + bd + (ab+bc) + (ad+cd)\right) \overset{\text{via } (1)}{=}$$

$$4s^2 - 4r^2 - 4r * \sqrt{4R^2 + r^2} - 2(bs + ds) = 4s^2 - 4r^2 - 4r * \sqrt{4R^2 + r^2} - 2s^2$$

$$\therefore \sum_{cyc} a^2 = 2s^2 - 4r^2 - 4r * \sqrt{4R^2 + r^2} \to (2)$$

Via Ptolemy's first and second theorems : $ef = ac + bd$ and $\frac{e}{f} = \frac{ad+bc}{ab+cd}$

$$\Rightarrow e^2 = \frac{(ac+bd)(ab+cd)(ad+bc)}{(ab+cd)^2} \overset{\text{Brahmagupta and Parameshvara}}{=} \frac{16R^2r^2s^2}{(ab+cd)^2}$$

$$\overset{CBS}{\ge} \frac{16R^2r^2s^2}{(a^2+c^2)(b^2+d^2)} \text{ and similarly, } f^2 = \frac{(ac+bd)(ab+cd)(ad+bc)}{(ad+bc)^2}$$

$$\overset{CBS}{\ge} \frac{16R^2r^2s^2}{(a^2+c^2)(b^2+d^2)} \therefore e^2, f^2 \ge \frac{16R^2r^2s^2}{(a^2+c^2)(b^2+d^2)} \therefore m^2 \ge \frac{16R^2r^2s^2}{(a^2+c^2)(b^2+d^2)}$$

$$= \frac{16R^2r^2s^2}{(s^2-2ac)(s^2-2bd)} = \frac{16R^2r^2s^2}{s^4 - 2s^2(ac+bd) + 4abcd} \overset{\text{via } (1)}{=}$$

$$\frac{16R^2r^2s^2}{s^4 - 2s^2\left(2r^2 + 2r * \sqrt{4R^2+r^2}\right) + 4r^2s^2} \Rightarrow \boxed{m^2 \overset{(*)}{\ge} \frac{16R^2r^2}{s^2 - 4r * \sqrt{4R^2 + r^2}}}$$

$$\text{Now, } (a^2+b^2+c^2+d^2)^3 \ge \frac{1}{4}\left(\sum_{cyc} a^2\right)^2\left(\sum_{cyc} a\right)^2 \overset{via\,(2)}{=}$$

$$\frac{4s^2}{4}\left((2s^2-4r^2)^2+16r^2(4R^2+r^2)-8(2s^2-4r^2)r*\sqrt{4R^2+r^2}\right)$$

$$\Rightarrow m^2(a^2+b^2+c^2+d^2)^3 \ge$$

$$\begin{pmatrix}(2s^2-4r^2)^2+16r^2(4R^2+r^2)\\ -8(2s^2-4r^2)r*\sqrt{4R^2+r^2}\end{pmatrix}\left(\frac{16R^2r^2}{s^2-4r*\sqrt{4R^2+r^2}}\right) \overset{?}{\ge} 64\alpha^2F^4$$

$$= 64R^2r^2s^4 \left(\because \frac{R}{r}=\alpha \text{ and } F=rs\right)$$

$$\Leftrightarrow (2s^2-4r^2)^2+16r^2(4R^2+r^2)-8(2s^2-4r^2)r*\sqrt{4R^2+r^2}$$

$$\overset{?}{\ge} 4s^2\left(s^2-4r*\sqrt{4R^2+r^2}\right)$$

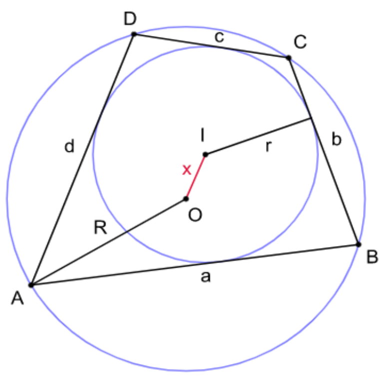
$$\Leftrightarrow 16r^2(4R^2+2r^2-s^2)+(16rs^2-16rs^2+32r^3)*\sqrt{4R^2+r^2} \overset{?}{\ge} 0$$

$$\Leftrightarrow s^2 \overset{?}{\le} 4R^2+2r^2+2r*\sqrt{4R^2+r^2} = \left(\sqrt{4R^2+r^2}+r\right)^2 \Leftrightarrow s \overset{?}{\le} \sqrt{4R^2+r^2}+r$$

$$\rightarrow \text{true via Blundon} - \text{Eddy} \therefore m^2(a^2+b^2+c^2+d^2)^3 \ge 64\alpha^2F^4$$

$$\Rightarrow a^2+b^2+c^2+d^2 \ge 4\sqrt[3]{\frac{\alpha^2}{m^2}}.F^{\frac{4}{3}}\ \forall \text{ bicentric quadrilateral ABCD (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco



We assume that $m = e$. We have

$$2F = ab\sin B + cd\sin D \text{ and } \sin B = \sin D = \frac{e}{2R},$$

$$\text{then } m = e = \frac{4RF}{ab+cd} \text{ or } \frac{\alpha}{m} = \frac{ab+cd}{4rF} = \frac{(a+b+c+d)(ab+cd)}{8F^2}.$$

$$\text{Now, since } (a+b+c+d)^2 \overset{CBS}{\le} 4(a^2+b^2+c^2+d^2) \text{ and } ab + cd \overset{AM-GM}{\le} \frac{a^2+b^2+c^2+d^2}{2}, \text{ then}$$

$$4\sqrt[3]{\frac{\alpha^2}{m^2}}.F^{\frac{4}{3}} = \sqrt[3]{(a+b+c+d)^2(ab+cd)^2}$$

$$\leq \sqrt[3]{4(a^2+b^2+c^2+d^2).\left(\frac{a^2+b^2+c^2+d^2}{2}\right)^2}$$

$$= a^2+b^2+c^2+d^2$$

which completes the proof. Equality holds iff $ABCD$ is square.