ROMANIAN MATHEMATICAL MAGAZINE

In any bicentric quadrilateral ABCD with sides → a, b, c, d, the following relationship holds :

$$\frac{2s+a}{4s-a}+\frac{2s+b}{4s-b}+\frac{2s+c}{4s-c}+\frac{2s+d}{4s-d}\ge\frac{40}{7}\left(\frac{r}{R}\right)^2$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\text{Via Brahmagupta and Parameshvara, } 16F^2R^2$$
$$=(ac+bd)(ab+cd)(ad+bc)$$
$$\Rightarrow x\left(bd\left((a+c)^2-2ac\right)+ac\left((b+d)^2-2bd\right)\right)=16R^2r^2s^2\ (x=ac+bd)$$
$$\Rightarrow x\left(bd(s^2-2ac)+ac(s^2-2bd)\right)=16R^2r^2s^2\Rightarrow x(s^2x-4r^2s^2)=16R^2r^2s^2$$
$$\Rightarrow x^2-4x.r^2-16R^2r^2=0\Rightarrow x=\frac{4r^2\pm\sqrt{64R^2r^2+16r^4}}{2}$$
$$\Rightarrow ac+bd=2r^2+2r*\sqrt{4R^2+r^2}\to(1)$$
$$\therefore\sum_{cyc}a^2=4s^2-2\left(ac+bd+(ab+bc)+(ad+cd)\right)\overset{\text{via (1)}}{=}$$
$$4s^2-4r^2-4r*\sqrt{4R^2+r^2}-2(bs+ds)=4s^2-4r^2-4r*\sqrt{4R^2+r^2}-2s^2$$
$$\therefore\sum_{cyc}a^2=2s^2-4r^2-4r*\sqrt{4R^2+r^2}\to(2)$$
$$\frac{2s+a}{4s-a}+\frac{2s+b}{4s-b}+\frac{2s+c}{4s-c}+\frac{2s+d}{4s-d}\ge\frac{40}{7}\left(\frac{r}{R}\right)^2\Leftrightarrow\sum_{cyc}\frac{4s-a+3a}{4s-a}\ge\frac{80}{7}\left(\frac{r}{R}\right)^2$$
$$\Leftrightarrow 4+3\sum_{cyc}\frac{a}{4s-a}\overset{(*)}{\ge}\frac{80}{7}\left(\frac{r}{R}\right)^2$$
$$\text{Again, }4+3\sum_{cyc}\frac{a}{4s-a}=4+3\sum_{cyc}\frac{a^2}{4sa-a^2}\overset{\text{Bergstrom}}{\ge}4+\frac{12s^2}{4s(2s)-\sum_{cyc}a^2}$$
$$\overset{?}{\ge}\frac{80}{7}\left(\frac{r}{R}\right)^2\Leftrightarrow(77R^2-160r^2)s^2\overset{?}{\ge}\left(\sum_{cyc}a^2\right)(7R^2-20r^2)$$
$$\overset{\text{via (2)}}{\Leftrightarrow}(77R^2-160r^2)s^2\overset{?}{\ge}(7R^2-20r^2)\left(2s^2-4r^2-4r*\sqrt{4R^2+r^2}\right)$$
$$\Leftrightarrow(63R^2-120r^2)s^2+(7R^2-20r^2)\left(4r^2+4r*\sqrt{4R^2+r^2}\right)\underbrace{\overset{?}{\ge}}_{(**)}0$$
$$\because 63R^2-120r^2=63(R^2-2r^2)+6r^2\overset{\text{L.Fejes Toth,1948}}{\ge}6r^2>0$$
$$\therefore\text{ via Blundon}-\text{Eddy, LHS of }(**)\ge(63R^2-120r^2)*8r*\left(\sqrt{4R^2+r^2}-r\right)$$
$$+(7R^2-20r^2)\left(4r^2+4r*\sqrt{4R^2+r^2}\right)\overset{?}{\ge}0$$
$$\Leftrightarrow\sqrt{4R^2+r^2}*(133R^2-260r^2)\underbrace{\overset{?}{\ge}}_{(***)}r(119R^2-220r^2)$$
$$\text{Once again, via L. Fejes Toth, LHS of }(***)\ge$$

$$3r(133R^2 - 260r^2) \overset{?}{\geq} r(119R^2 - 220r^2) \Leftrightarrow 280R^2 \overset{?}{\geq} 560r^2$$
$$\rightarrow \text{true, via } L. \text{ Fejes Toth} \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$
$$\therefore \frac{2s+a}{4s-a} + \frac{2s+b}{4s-b} + \frac{2s+c}{4s-c} + \frac{2s+d}{4s-d} \geq \frac{40}{7}\left(\frac{r}{R}\right)^2$$
$$\forall \text{ bicentric quadrilateral } ABCD \text{ (QED)}$$