ROMANIAN MATHEMATICAL MAGAZINE

In any bicentric quadrilateral ABCD with sides a, b, c, d such that $a \le b \le c \le d$, the following relationship holds:

$$\sqrt[4]{\frac{a}{r}}+\sqrt[4]{\frac{b}{r}}+\sqrt[4]{\frac{c}{r}} \le 3.\sqrt{\frac{R}{r}}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$a \le b \le c \le d \Rightarrow b \le d \Rightarrow b \le \frac{b+d}{2} \Rightarrow \sqrt[4]{\frac{a}{r}}+\sqrt[4]{\frac{b}{r}}+\sqrt[4]{\frac{c}{r}}$$

$$\le \frac{1}{\sqrt[4]{r}}\left(\sqrt[4]{a}+\sqrt[4]{c}\right)+\sqrt[4]{\frac{b+d}{2r}} \overset{CBS}{\le} \frac{\sqrt{2}}{\sqrt[4]{r}}.\sqrt{\sqrt{a}+\sqrt{c}}+\sqrt[4]{\frac{b+d}{2r}} \overset{CBS}{\le}$$

$$\frac{\sqrt{2}}{\sqrt[4]{r}}.\sqrt{\sqrt{2}.\sqrt{a+c}}+\sqrt[4]{\frac{b+d}{2r}} = 2.\sqrt[4]{\frac{a+c}{2r}}+\sqrt[4]{\frac{b+d}{2r}} = 2.\sqrt[4]{\frac{s}{2r}}+\sqrt[4]{\frac{s}{2r}} = 3.\sqrt[4]{\frac{s}{2r}}$$

$$\overset{?}{\le} 3.\sqrt{\frac{R}{r}} \Leftrightarrow r^2s^2 \underbrace{\overset{?}{\le}}_{(*)} 4R^4$$

$$\text{Now, via Blundon – Eddy, } s \le \sqrt{4R^2+r^2}+r$$

$$\Rightarrow r^2s^2 \le r^2\left(4R^2+2r^2+2r.\sqrt{4R^2+r^2}\right) \overset{?}{\le} 4R^4$$

$$\Leftrightarrow 4R^4-4R^2r^2-2r^4 \overset{?}{\ge} 2r^3.\sqrt{4R^2+r^2}$$

$$\Leftrightarrow \left(4R^4-4R^2r^2-2r^4\right)^2 \overset{?}{\ge} 4r^6.\left(4R^2+r^2\right)$$

$$\left(\because 4R^4-4R^2r^2-2r^4=\left(4R^2+4r^2\right)\left(R^2-2r^2\right)+6r^4 \overset{\text{L.Fejes Toth}}{\ge} 6r^4>0\right)$$

$$\Leftrightarrow 16R^6\left(R^2-2r^2\right) \overset{?}{\ge} 0 \to \text{true, via L. Fejes Toth} \Rightarrow (*) \text{ is true}$$

$$\therefore \sqrt[4]{\frac{a}{r}}+\sqrt[4]{\frac{b}{r}}+\sqrt[4]{\frac{c}{r}} \le 3.\sqrt{\frac{R}{r}}$$

$\forall$ bicentric quadrilateral ABCD with sides $a, b, c, d \mid a \le b \le c \le d$ (QED)