ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{(b^2+c^2)\frac{\sin C}{\sin A}+(c^2+a^2)\frac{\sin B}{\sin A}}{\sqrt{\frac{b^2+c^2}{a^2+b^2}}.\sin C+\sqrt{\frac{c^2+a^2}{a^2+b^2}}.\sin B}\ge 48\sqrt{3}r^2$$

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$$\text{Firstly, }\sqrt[3]{\left(\sum_{cyc}a^2+a^2\right)\left(\sum_{cyc}a^2+b^2\right)\left(\sum_{cyc}a^2+c^2\right)}$$

$$\overset{\text{Leibnitz}}{\le}\sqrt[3]{(9R^2+a^2)(9R^2+b^2)(9R^2+c^2)}\overset{A-G}{\le}\sum_{cyc}\frac{9R^2+a^2}{3}=\frac{27R^2+\sum_{cyc}a^2}{3}$$

$$\overset{\text{Leibnitz}}{\le}\frac{27R^2+9R^2}{3}=12R^2\Rightarrow\left(\sum_{cyc}a^2+a^2\right)\left(\sum_{cyc}a^2+b^2\right)\left(\sum_{cyc}a^2+c^2\right)\le$$

$$144*12R^6\Rightarrow\sqrt{\left(\sum_{cyc}a^2+a^2\right)\left(\sum_{cyc}a^2+b^2\right)\left(\sum_{cyc}a^2+c^2\right)}\le 24\sqrt{3}R^3\to(1)$$

$$\text{Now, }\frac{(b^2+c^2)\frac{\sin C}{\sin A}+(c^2+a^2)\frac{\sin B}{\sin A}}{\sqrt{\frac{b^2+c^2}{a^2+b^2}}.\sin C+\sqrt{\frac{c^2+a^2}{a^2+b^2}}.\sin B}+\frac{(a^2+b^2)\frac{\sin C}{\sin B}+(c^2+a^2)\frac{\sin A}{\sin B}}{\sqrt{\frac{a^2+b^2}{b^2+c^2}}.\sin C+\sqrt{\frac{c^2+a^2}{b^2+c^2}}.\sin A}$$

$$+\frac{(a^2+b^2)\frac{\sin B}{\sin C}+(b^2+c^2)\frac{\sin A}{\sin C}}{\sqrt{\frac{a^2+b^2}{c^2+a^2}}.\sin B+\sqrt{\frac{b^2+c^2}{c^2+a^2}}.\sin A}$$

$$\overset{CBS}{\ge}2R.\frac{(b^2+c^2)\frac{c}{a}+(c^2+a^2)\frac{b}{a}}{\sqrt{\frac{b^2+c^2}{a^2+b^2}+\frac{c^2+a^2}{a^2+b^2}}.\sqrt{b^2+c^2}}+2R.\frac{(a^2+b^2)\frac{c}{b}+(c^2+a^2)\frac{a}{b}}{\sqrt{\frac{a^2+b^2}{b^2+c^2}+\frac{c^2+a^2}{b^2+c^2}}.\sqrt{c^2+a^2}}$$

$$+2R.\frac{(a^2+b^2)\frac{b}{c}+(b^2+c^2)\frac{a}{c}}{\sqrt{\frac{a^2+b^2}{c^2+a^2}+\frac{b^2+c^2}{c^2+a^2}}.\sqrt{a^2+b^2}}\overset{A-G}{\ge}4R.\frac{\sqrt{(b^2+c^2)(c^2+a^2)}.\sqrt{\frac{bc}{a^2}}.\sqrt{a^2+b^2}}{\sqrt{b^2+c^2}.\sqrt{\sum_{cyc}a^2+c^2}}+$$

$$4R.\frac{\sqrt{(a^2+b^2)(c^2+a^2)}.\sqrt{\frac{ca}{b^2}}.\sqrt{b^2+c^2}}{\sqrt{c^2+a^2}.\sqrt{\sum_{cyc}a^2+a^2}}+4R.\frac{\sqrt{(a^2+b^2)(b^2+c^2)}.\sqrt{\frac{ab}{c^2}}.\sqrt{c^2+a^2}}{\sqrt{a^2+b^2}.\sqrt{\sum_{cyc}a^2+b^2}}$$

$$\overset{A-G}{\geq} 12R.\sqrt[3]{\frac{(a^2+b^2)(b^2+c^2)(c^2+a^2)}{\sqrt{\left(\sum_{cyc} a^2+a^2\right)\left(\sum_{cyc} a^2+b^2\right)\left(\sum_{cyc} a^2+c^2\right)}}} \overset{\substack{Cesaro\\ and\\ via\ (1)}}{\geq} 12R.\sqrt[3]{\frac{8a^2b^2c^2}{24\sqrt{3}R^3}}$$

$$= \frac{12R}{\sqrt{3}R}.\sqrt[3]{16R^2r^2s^2} \overset{\substack{Euler\\ and\\ Mitrinovic}}{\geq} \frac{12}{\sqrt{3}}.\sqrt[3]{64*27r^6} = \frac{48}{\sqrt{3}}.3r^2$$

$$\therefore \frac{(b^2+c^2)\frac{\sin C}{\sin A}+(c^2+a^2)\frac{\sin B}{\sin A}}{\sqrt{\frac{b^2+c^2}{a^2+b^2}}.\sin C+\sqrt{\frac{c^2+a^2}{a^2+b^2}}.\sin B}+\frac{(a^2+b^2)\frac{\sin C}{\sin B}+(c^2+a^2)\frac{\sin A}{\sin B}}{\sqrt{\frac{a^2+b^2}{b^2+c^2}}.\sin C+\sqrt{\frac{c^2+a^2}{b^2+c^2}}.\sin A}$$

$$+\frac{(a^2+b^2)\frac{\sin B}{\sin C}+(b^2+c^2)\frac{\sin A}{\sin C}}{\sqrt{\frac{a^2+b^2}{c^2+a^2}}.\sin B+\sqrt{\frac{b^2+c^2}{c^2+a^2}}.\sin A} \geq 48\sqrt{3}r^2\ \forall\ \Delta\ ABC,$$

$'' ='' $ iff Δ ABC is equilateral (QED)