In any Δ ABC, the following relationship holds :

$$\sqrt{1+\frac{R}{2r}\left(\frac{h_a+h_b+h_c}{s}\right)^2} \ge \sqrt{\frac{h_b+h_c}{h_a+h_c}}+\sqrt{\frac{h_a+h_c}{h_b+h_c}}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India, Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sqrt{1+\frac{R}{2r}\left(\frac{h_a+h_b+h_c}{s}\right)^2} \ge \sqrt{\frac{h_b+h_c}{h_a+h_c}}+\sqrt{\frac{h_a+h_c}{h_b+h_c}}$$

$$\Leftrightarrow 1+\frac{R}{2r}\cdot\frac{\left(\sum_{cyc} ab\right)^2}{4R^2s^2} \ge 2+\frac{ca+ab}{bc+ab}+\frac{bc+ab}{ca+ab}$$

$$\Leftrightarrow \frac{(ab+bc+ca)^2}{abc(a+b+c)} \ge \frac{(ca+ab)^2+(bc+ab)^2+(ca+ab)(bc+ab)}{(ca+ab)(bc+ab)}$$

$$\Leftrightarrow (ca+ab)(bc+ab)(ab+bc+ca)^2$$

$$-abc(a+b+c)\big((ca+ab)^2+(bc+ab)^2+(ca+ab)(bc+ab)\big) \ge 0$$

$$\Leftrightarrow a^4b^4+a^2b^2c^4-2a^3b^3c^2 \ge 0 \Leftrightarrow a^2b^2(ab-c^2)^2 \ge 0 \to \text{true}$$

$$\therefore \sqrt{1+\frac{R}{2r}\left(\frac{h_a+h_b+h_c}{s}\right)^2} \ge \sqrt{\frac{h_b+h_c}{h_a+h_c}}+\sqrt{\frac{h_a+h_c}{h_b+h_c}}\ \forall\,\Delta\,ABC\ (QED)$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Lemma : In ΔABC, we have

$$\frac{b}{c}+\frac{c}{b} \le \frac{\sqrt{a^2b^2+b^2c^2+c^2a^2}}{2F} \qquad (*)$$

Proof : Since $16F^2 = 2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)$, then we have

$$(*) \Leftrightarrow (b^2+c^2)\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}$$
$$\le 2bc\sqrt{a^2b^2+b^2c^2+c^2a^2}$$

$$\overset{squaring}{\Leftrightarrow} \quad (2b^2c^2 + b^4 + c^4)[2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)] \leq 4b^2c^2(a^2b^2 + b^2c^2 + c^2a^2)$$

$$\Leftrightarrow \; -a^4(b^2 + c^2)^2 + 2(b^4 + c^4)(a^2b^2 + c^2a^2) - (b^4 + c^4)^2 = -[a^2(b^2 + c^2) - (b^4 + c^4)]^2 \leq 0,$$

which is true and the proof of the lemma is complete.

Now, since $\sqrt{h_a + h_b}, \sqrt{h_b + h_c}, \sqrt{h_a + h_c}$ **can be the sides of triangle with area** F' **such that**

$$16F'^2 = 2\sum_{cyc} \sqrt{h_a + h_b}^2 \sqrt{h_a + h_c}^2 - \sum_{cyc} \sqrt{h_b + h_c}^4 = 4\sum_{cyc} h_b h_c = \frac{8s^2r}{R}.$$

Then by using the lemma in this triangle, we obtain

$$\frac{\sqrt{h_b + h_c}}{\sqrt{h_a + h_c}} + \frac{\sqrt{h_a + h_c}}{\sqrt{h_b + h_c}} \leq \sqrt{\frac{\sum_{cyc} \sqrt{h_a + h_b}^2 \sqrt{h_a + h_c}^2}{4F'^2}} = \sqrt{\frac{\sum_{cyc} h_b h_c}{4F'^2} + \frac{\left(\sum_{cyc} h_a\right)^2}{4F'^2}}$$

$$= \sqrt{1 + \frac{R}{2r}\left(\frac{h_a + h_b + h_c}{s}\right)^2},$$

which completes the proof. Equality hlods iff ΔABC **is equilateral.**