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In ΔABC the following relationship holds:

$$\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}+\frac{\sin^2\frac{B}{2}}{\sin^2\frac{A}{2}}\geq\frac{r_a}{r_b}+\frac{r_b}{r_a}$$

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We have

$$\left(\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}+\frac{\sin^2\frac{B}{2}}{\sin^2\frac{A}{2}}\right)-\left(\frac{r_a}{r_b}+\frac{r_b}{r_a}\right)=\frac{a(s-b)}{b(s-a)}+\frac{b(s-a)}{a(s-b)}-\frac{s-b}{s-a}-\frac{s-a}{s-b}$$

$$=\frac{s-b}{s-a}\left(\frac{a}{b}-1\right)-\frac{s-a}{s-b}\left(1-\frac{b}{a}\right)=\frac{(a-b)(s-b)}{b(s-a)}-\frac{(a-b)(s-a)}{a(s-b)}$$

$$=\frac{(a-b)[a(s-b)^2-b(s-a)^2]}{ab(s-a)(s-b)}=\frac{(a-b)^2(s^2-ab)}{ab(s-a)(s-b)}\overset{s>a,b}{\geq}0,$$

Therefore

$$\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}+\frac{\sin^2\frac{B}{2}}{\sin^2\frac{A}{2}}\geq\frac{r_a}{r_b}+\frac{r_b}{r_a}.$$

Equality holds iff $a=b$.