ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the $following$ relationship ho$l$ds :

$$5+\sum_{cyc}\frac{n_b n_c}{r_b r_c}\le\frac{4R}{r}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $n_a{}^2=s(s-a)+\frac{s(b-c)^2}{a}=s^2-\frac{s[a^2-(b-c)^2]}{a}=s^2-\frac{4s(s-b)(s-c)}{a}$

$$=s^2-\frac{4s.sr^2}{a(s-a)}=s^2-2h_a r_a\ (\text{and analogs})\ \text{ and }\ \frac{1}{r_b}+\frac{1}{r_c}=\frac{2}{h_a}\ (\text{and analogs}).$$

Using these identities, we have

$$\sum_{cyc}\frac{n_b n_c}{r_b r_c}\overset{AM-GM}{\le}\sum_{cyc}\frac{n_b{}^2+n_c{}^2}{2r_b r_c}=\sum_{cyc}\frac{(s^2-2h_b r_b)+(s^2-2h_c r_c)}{2r_b r_c}=\sum_{cyc}\left(\frac{s^2}{r_b r_c}-\left(\frac{h_b}{r_c}+\frac{h_c}{r_b}\right)\right)$$

$$=\frac{s^2(r_a+r_b+r_c)}{r_a r_b r_c}-\sum_{cyc}\left(\frac{h_a}{r_b}+\frac{h_a}{r_c}\right)=\frac{s^2(4R+r)}{s^2 r}-\sum_{cyc}2=\frac{4R}{r}-5,$$

which completes the proof. Equality holds iff ΔABC is equilateral.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$\text{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2$$

$$\Rightarrow an_a^2=as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}$$
$$=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)$$
$$=as^2-2ah_a r_a\Rightarrow h_a r_a=\frac{s^2-n_a^2}{2}\ \textit{and analogs}\Rightarrow n_b n_c+h_b r_b+h_c r_c$$
$$=n_b n_c+\frac{s^2-n_b^2}{2}+\frac{s^2-n_c^2}{2}=\frac{2s^2-(n_b^2+n_c^2-n_b n_c)}{2}=\frac{2s^2-(n_b-n_c)^2}{2}\le\frac{2s^2}{2}$$

$$\Rightarrow \frac{n_b n_c + h_b r_b + h_c r_c}{r_b r_c} \leq \frac{s^2}{s(s-a)} \Rightarrow \frac{n_b n_c}{r_b r_c} \overset{(*)}{\leq} \frac{s}{s-a} - \frac{h_b}{r_c} - \frac{h_c}{r_b} \text{ and analogously,}$$

$$\frac{n_c n_a}{r_c r_a} \overset{(**)}{\leq} \frac{s}{s-b} - \frac{h_c}{r_a} - \frac{h_a}{r_c} \text{ and } \frac{n_a n_b}{r_a r_b} \overset{(***)}{\leq} \frac{s}{s-c} - \frac{h_a}{r_b} - \frac{h_b}{r_a} \therefore (*) + (**) + (***) \Rightarrow$$

$$\sum_{cyc} \frac{n_b n_c}{r_b r_c} \leq \sum_{cyc} \frac{s}{s-a} - \sum_{cyc} \frac{h_a(r_b + r_c)}{r_b r_c} \overset{\text{via (i) and analogs}}{=} \frac{s}{r^2 s} \cdot \sum_{cyc} (s-b)(s-c)$$

$$- \sum_{cyc} \frac{\frac{bc}{2R} \cdot 4R\cos^2\frac{A}{2}}{bc \cdot \cos^2\frac{A}{2}} = \frac{4Rr + r^2}{r^2} - 6 \Rightarrow 5 + \sum_{cyc} \frac{n_b n_c}{r_b r_c} \leq \frac{4R}{r} + 1 - 6 + 5 = \frac{4R}{r}$$

$\forall \Delta ABC$, " $=$ " iff ΔABC is equilateral (QED)