ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta$ ABC, the following relationship holds :

$$\sum_{cyc}\sqrt{\frac{1}{2}\left(\frac{n_bn_c}{r_br_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}-1\right)}\le\sqrt{\frac{2R-r+AI+BI+CI}{r}}$$

*Proposed by Bogdan Fuștei-Romania*

## Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $n_a{}^2=s(s-a)+\frac{s(b-c)^2}{a}=s^2-\frac{s[a^2-(b-c)^2]}{a}=s^2-\frac{4s(s-b)(s-c)}{a}$

$$=s^2-\frac{4s.sr^2}{a(s-a)}=s^2-2h_ar_a\text{ (and analogs)},$$

then

$$\frac{n_bn_c}{r_br_c}\overset{AM-GM}{\le}\frac{n_b{}^2+n_c{}^2}{2r_br_c}=\frac{(s^2-2h_br_b)+(s^2-2h_cr_c)}{2r_br_c}=\frac{s^2}{r_br_c}-\left(\frac{h_b}{r_c}+\frac{h_c}{r_b}\right)\text{ (and analogs)}.$$

Using this inequality, we have

$$\sum_{cyc}\sqrt{\frac{1}{2}\left(\frac{n_bn_c}{r_br_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}-1\right)}\le\sum_{cyc}\sqrt{\frac{1}{2}\left(\frac{s^2}{r_br_c}-1\right)}=\sum_{cyc}\sqrt{\frac{1}{2}\left(\frac{s}{s-a}-1\right)}=\sum_{cyc}\sqrt{\frac{a}{2(s-a)}}$$

$$=\sqrt{\sum_{cyc}\frac{a}{2(s-a)}+2\sum_{cyc}\sqrt{\frac{bc}{2(s-b).2(s-c)}}}=\sqrt{\frac{2R-r}{r}+\sum_{cyc}\frac{1}{\sin\frac{A}{2}}}=\sqrt{\frac{2R-r+AI+BI+CI}{r}},$$

as desired. Equality holds iff $\Delta ABC$ is equilateral.

## Solution 2 by Soumava Chakraborty-Kolkata-India

$$:\text{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2$$

$$\Rightarrow an_a^2=as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}$$
$$=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)$$
$$=as^2-2ah_ar_a\Rightarrow h_ar_a=\frac{s^2-n_a^2}{2}\text{ and analogs}\Rightarrow n_bn_c+h_br_b+h_cr_c$$
$$=n_bn_c+\frac{s^2-n_b^2}{2}+\frac{s^2-n_c^2}{2}=\frac{2s^2-(n_b^2+n_c^2-n_bn_c)}{2}=\frac{2s^2-(n_b-n_c)^2}{2}\le\frac{2s^2}{2}$$

$$\Rightarrow \frac{n_b n_c + h_b r_b + h_c r_c}{r_b r_c} \le \frac{s^2}{s(s-a)}$$

$$\Rightarrow \sqrt{\frac{1}{2}\left(\frac{n_b n_c}{r_b r_c} + \frac{h_b}{r_c} + \frac{h_c}{r_b} - 1\right)} \le \sqrt{\frac{1}{2}\left(\frac{s^2}{s(s-a)} - 1\right)} = \sqrt{\frac{1}{2} \cdot \frac{a}{s-a}}$$

$$= \sqrt{\frac{1}{2} \cdot \frac{4R\cos\frac{A}{2}\sin\frac{A}{2}}{4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}} = \sqrt{\frac{1}{2} \cdot \frac{\sin^2\frac{A}{2}}{\frac{r}{4R}}}$$

$$\Rightarrow \sqrt{\frac{1}{2}\left(\frac{n_b n_c}{r_b r_c} + \frac{h_b}{r_c} + \frac{h_c}{r_b} - 1\right)} \le \sqrt{\frac{2R}{r}} . \sin\frac{A}{2} \textit{ and analogs}$$

$$\Rightarrow \sum_{cyc} \sqrt{\frac{1}{2}\left(\frac{n_b n_c}{r_b r_c} + \frac{h_b}{r_c} + \frac{h_c}{r_b} - 1\right)} \le \sqrt{\frac{2R}{r}} . \sum_{cyc} \sin\frac{A}{2} \overset{?}{\le} \sqrt{\frac{2R - r + AI + BI + CI}{r}}$$

$$\Leftrightarrow \sum_{cyc} \sin\frac{A}{2} \overset{?}{\le} \sqrt{\frac{2R - r + AI + BI + CI}{2R}} = \sqrt{\frac{2R - r}{2R} + \sum_{cyc}\left(\frac{r}{2R} \cdot \frac{1}{\sin\frac{A}{2}}\right)}$$

$$= \sqrt{\sum_{cyc} \sin^2\frac{A}{2} + \sum_{cyc}\left(2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \cdot \frac{1}{\sin\frac{A}{2}}\right)} = \sqrt{\sum_{cyc} \sin^2\frac{A}{2} + 2\sum_{cyc} \sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\Leftrightarrow \sum_{cyc} \sin\frac{A}{2} \overset{?}{\le} \sqrt{\left(\sum_{cyc} \sin\frac{A}{2}\right)^2} \to \text{true} \therefore \sum_{cyc} \sqrt{\frac{1}{2}\left(\frac{n_b n_c}{r_b r_c} + \frac{h_b}{r_c} + \frac{h_c}{r_b} - 1\right)}$$

$$\le \sqrt{\frac{2R - r + AI + BI + CI}{r}} \ \forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$