ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\prod_{cyc}\left(\frac{n_b n_c}{r_b r_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}\right)\le\frac{s^2}{r^2}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $n_a{}^2=s(s-a)+\frac{s(b-c)^2}{a}=s^2-\frac{s[a^2-(b-c)^2]}{a}=s^2-\frac{4s(s-b)(s-c)}{a}$

$$=s^2-\frac{4s.sr^2}{a(s-a)}=s^2-2h_a r_a \text{ (and analogs)},$$

then

$$\frac{n_b n_c}{r_b r_c}\overset{AM-GM}{\le}\frac{n_b{}^2+n_c{}^2}{2r_b r_c}=\frac{(s^2-2h_b r_b)+(s^2-2h_c r_c)}{2r_b r_c}=\frac{s^2}{r_b r_c}-\left(\frac{h_b}{r_c}+\frac{h_c}{r_b}\right) \text{ (and analogs).}$$

Using this inequality, we have

$$\prod_{cyc}\left(\frac{n_b n_c}{r_b r_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}\right)\le\prod_{cyc}\left(\frac{s^2}{r_b r_c}\right)=\frac{(s^2)^3}{(s^2 r)^2}=\frac{s^2}{r^2},$$

as desired. Equality holds iff ΔABC is equilateral.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2$$
$$\Rightarrow an_a^2=as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}$$
$$=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)$$
$$=as^2-2ah_a r_a\Rightarrow h_a r_a=\frac{s^2-n_a^2}{2}\text{ and analogs}\Rightarrow n_b n_c+h_b r_b+h_c r_c$$
$$=n_b n_c+\frac{s^2-n_b^2}{2}+\frac{s^2-n_c^2}{2}=\frac{2s^2-(n_b^2+n_c^2-n_b n_c)}{2}=\frac{2s^2-(n_b-n_c)^2}{2}\le\frac{2s^2}{2}$$
$$\Rightarrow\frac{n_b n_c+h_b r_b+h_c r_c}{r_b r_c}\le\frac{s^2}{s(s-a)}\Rightarrow\frac{n_b n_c}{r_b r_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}\le\frac{s}{s-a}\text{ and analogs}$$
$$\Rightarrow\prod_{cyc}\left(\frac{n_b n_c}{r_b r_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}\right)\le\frac{s^3}{(s-a)(s-b)(s-c)}=\frac{s^3}{r^2 s}$$
$$\therefore\prod_{cyc}\left(\frac{n_b n_c}{r_b r_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}\right)\le\frac{s^2}{r^2}\ \forall\,\Delta\,ABC,\ ''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$