# ROMANIAN MATHEMATICAL MAGAZINE

**In any $\Delta$ ABC, the following relationships hold :**

$$\sqrt{r_b r_c}\left(\frac{1}{m_b}+\frac{1}{m_c}\right)\le \csc\frac{A}{2}\ \textit{and}\ \left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\left(\frac{1}{m_b}+\frac{1}{m_c}\right)\le 2\left(\frac{1}{h_b}+\frac{1}{h_c}\right)$$

*Proposed by Bogdan Fuștei-Romania*

*Solution 1 by Soumava Chakraborty-Kolkata-India*

$$\sqrt{r_b r_c}\left(\frac{1}{m_b}+\frac{1}{m_c}\right)\overset{\text{Lascu + A-G}}{\le}\sqrt{s(s-a)}\left(\frac{1}{\sqrt{s(s-b)}}+\frac{1}{\sqrt{s(s-c)}}\right)\overset{?}{\le}\csc\frac{A}{2}$$

$$\Leftrightarrow\sqrt{\frac{s-a}{s-b}}+\sqrt{\frac{s-a}{s-c}}\overset{?}{\le}\frac{\sqrt{bc}}{\sqrt{(s-b)(s-c)}}$$

$$\Leftrightarrow\sqrt{(s-a)(s-c)}+\sqrt{(s-a)(s-b)}\overset{?}{\le}\sqrt{bc}\Leftrightarrow\sqrt{xz}+\sqrt{xy}\underset{(*)}{\overset{?}{\le}}\sqrt{(z+x)(x+y)}$$

$$\left(\text{where } s-a=x, s-b=y, s-c=z\Rightarrow s=x+y+z\Rightarrow a=y+z, b=z+x,\ c=x+y\right)$$

$$\text{Now, }(z+x)(x+y)\overset{\text{Reverse CBS}}{\ge}\left(\sqrt{xz}+\sqrt{xy}\right)^2\Rightarrow(*)\text{ is true}$$

$$\therefore\ \boxed{\sqrt{r_b r_c}\left(\frac{1}{m_b}+\frac{1}{m_c}\right)\le\csc\frac{A}{2}}\Rightarrow\frac{1}{m_b}+\frac{1}{m_c}\le\frac{\sqrt{bc}}{\sqrt{(s-b)(s-c)}.\sqrt{s(s-a)}}$$

$$\Rightarrow\frac{1}{m_b}+\frac{1}{m_c}\le\frac{\sqrt{bc}}{F}\to(1)$$

$$\text{Now, }\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\left(\frac{1}{m_b}+\frac{1}{m_c}\right)\le 2\left(\frac{1}{h_b}+\frac{1}{h_c}\right)\Leftrightarrow\left(\frac{b+c}{\sqrt{bc}}\right)\left(\frac{1}{m_b}+\frac{1}{m_c}\right)\le 2.\frac{b+c}{2F}$$

$$\Leftrightarrow\frac{1}{m_b}+\frac{1}{m_c}\le\frac{\sqrt{bc}}{F}\to\text{true via }(1)\ \therefore\ \boxed{\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\left(\frac{1}{m_b}+\frac{1}{m_c}\right)\le 2\left(\frac{1}{h_b}+\frac{1}{h_c}\right)}$$

$$\therefore\ \sqrt{r_b r_c}\left(\frac{1}{m_b}+\frac{1}{m_c}\right)\le\csc\frac{A}{2}\ \textit{and}\ \left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\left(\frac{1}{m_b}+\frac{1}{m_c}\right)\le 2\left(\frac{1}{h_b}+\frac{1}{h_c}\right),$$

$$''='' \text{ iff }\Delta\text{ ABC is equilateral (QED)}$$

*Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco*

By Tereshin and CBS inequalities, we have

$$\sqrt{m_b m_c}\ge\sqrt{\frac{(c^2+a^2)(a^2+b^2)}{4R.4R}}\ge\frac{ac+ab}{4R}=\frac{abc}{4R}\left(\frac{1}{b}+\frac{1}{c}\right)=F\left(\frac{1}{b}+\frac{1}{c}\right).$$

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**We know that $m_a, m_b, m_c$ can be the sides of a triangle with area $F' = \frac{3F}{4}$ and medians $m'_a = \frac{3a}{4}$**

**$m'_b = \frac{3b}{4}, m'_c = \frac{3c}{4}$. By using the last inequality in this triangle, we have**

$$\sqrt{m'_b m'_c} \geq F'\left(\frac{1}{m_b} + \frac{1}{m_c}\right) \Rightarrow \frac{1}{m_b} + \frac{1}{m_c} \leq \frac{\sqrt{bc}}{F}.$$

**Using this result, we have**

$$\sqrt{r_b r_c}\left(\frac{1}{m_b} + \frac{1}{m_c}\right) \leq \sqrt{\frac{F^2}{(s-b)(s-b)}} \cdot \frac{\sqrt{bc}}{F} = \csc\frac{A}{2},$$

$$\left(\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}}\right)\left(\frac{1}{m_b} + \frac{1}{m_c}\right) \leq \frac{b+c}{\sqrt{bc}} \cdot \frac{\sqrt{bc}}{F} = 2\left(\frac{1}{h_b} + \frac{1}{h_c}\right),$$

**as desired. Equality holds iff $\Delta ABC$ is equilateral.**