ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$(x+y+z)(xAP^2+yBP^2+zCP^2)\geq yza^2+zxb^2+xyc^2+(x+y+z)^2{h_p}^2$$

for P a point in space,
h_p perpendicular from P on (ABC), x,y,z real numbers, a,b,c sides of ΔABC

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Let M be the feet of the perpendicular from P to (ABC).

We have $\left(x\overrightarrow{AM}+y\overrightarrow{BM}+z\overrightarrow{CM}\right)^2\geq 0$

$$\Leftrightarrow x^2AM^2+y^2BM^2+z^2CM^2+2xy\overrightarrow{AM}.\overrightarrow{BM}+2yz\overrightarrow{BM}.\overrightarrow{CM}+2zx\overrightarrow{CM}.\overrightarrow{AM}\geq 0.$$

Since $2\overrightarrow{BM}.\overrightarrow{CM}=BM^2+CM^2-BC^2$ (and analogs), and

$$AM^2=AP^2-{h_p}^2 \text{ (and analogs), then}$$

$$x^2\left(AP^2-{h_p}^2\right)+y^2\left(BP^2-{h_p}^2\right)+z^2\left(CP^2-{h_p}^2\right)+xy\left(AP^2+BP^2-2{h_p}^2-c^2\right)+$$

$$+yz\left(BP^2+CP^2-2{h_p}^2-a^2\right)+zx\left(CP^2+AP^2-2{h_p}^2-b^2\right)\geq 0$$

$$\Leftrightarrow (x+y+z)(xAP^2+yBP^2+zCP^2)\geq yza^2+zxb^2+xyc^2+(x+y+z)^2{h_p}^2,$$

as desired. Equality holds if and only if the barycentric coordinates of point M is (x,y,z).