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In any Δ ABC, the following relationship holds :

$$m_b + m_c \geq \sqrt{h_a^2 + \frac{9a^2}{4}}$$

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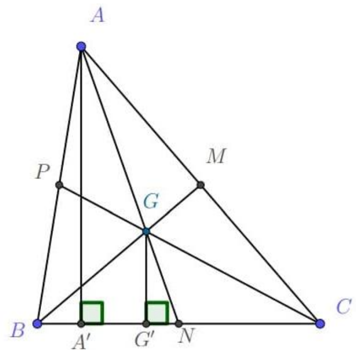
Solution 1 by Eric Cismaru-Romania



Let G be the centroid of triangle ΔABC and let M, N, P be the midpoints of sides AC, BC, AB. Construct $GG' \perp BC$ and $AA' \perp BC$.

Because $AA' \parallel GG' \Leftrightarrow \Delta GG'P \sim \Delta AA'P \Rightarrow GG' = \frac{h_a}{3}$.

Let's apply Pythagoras Theorem in triangles $\Delta BGG'$ and $\Delta GG'C$.

We obtain that $GB^2 = \frac{4m_b^2}{9} = \frac{h_a^2}{9} + BG'^2$ and $GC^2 = \frac{4m_c^2}{9} = \frac{h_a^2}{9} + CG'^2$ and by taking the square root and adding we'll have

$$\frac{2}{3}(m_b + m_c) = \sqrt{\frac{h_a^2}{9} + BG'^2} + \sqrt{\frac{h_a^2}{9} + BG'^2} = \sqrt{\left(\frac{h_a}{3}\right)^2 + BG'^2} + \sqrt{\left(\frac{h_a}{3}\right)^2 + CG'^2}$$

Using Minkowski's Inequality,

$$\frac{2}{3}(m_b + m_c) \geq \sqrt{\left(\frac{2h_a}{3}\right)^2 + (BG' + CG')^2} = \sqrt{\frac{4h_a^2}{9} + a^2},$$

which is equivalent to $m_b + m_c \geq \sqrt{h_a^2 + \frac{9a^2}{4}}$.

Equality holds when $BG' = CG'$ or when G' is the midpoint of BC, meaning that $G' = N \Leftrightarrow AN \perp BC \Leftrightarrow AB = AC$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$m_b^2+m_c^2=\frac{2c^2+2a^2-b^2+2a^2+2b^2-c^2}{4}=\frac{1}{4}\sum_{cyc}a^2+\frac{3a^2}{4}$$

$$\Rightarrow\frac{9a^2}{4}=3\left(m_b^2+m_c^2\right)-\frac{3}{4}\sum_{cyc}a^2=3\left(m_b^2+m_c^2\right)-\sum_{cyc}m_a^2$$

$$\Rightarrow\frac{9a^2}{4}=2\left(m_b^2+m_c^2\right)-m_a^2\rightarrow(1)$$

$$\therefore m_b+m_c\ge\sqrt{h_a^2+\frac{9a^2}{4}}\overset{\text{via }(1)}{\Leftrightarrow}m_b^2+m_c^2+2m_bm_c\ge h_a^2+2\left(m_b^2+m_c^2\right)-m_a^2$$

$$\Leftrightarrow\boxed{m_a^2-h_a^2\overset{(*)}{\ge}(m_b-m_c)^2}$$

$$\text{Now, }m_a^2-h_a^2=s(s-a)+\frac{(b-c)^2}{4}-\frac{4s(s-a)(s-b)(s-c)}{a^2}$$

$$=s(s-a)+\frac{(b-c)^2}{4}-\frac{s(s-a)\left(a^2-(b-c)^2\right)}{a^2}=\frac{(b-c)^2}{4}+\frac{s(s-a)(b-c)^2}{a^2}$$

$$=\frac{(b-c)^2}{4a^2}\left(a^2+4s^2-4sa\right)=\frac{(b-c)^2(2s-a)^2}{4a^2}=\frac{(b-c)^2(b+c)^2}{4a^2}$$

$$\Rightarrow m_a^2-h_a^2=\frac{\left(b^2-c^2\right)^2}{4a^2}\rightarrow(2)\therefore\text{via }(2),(*)\Leftrightarrow\boxed{\frac{\left(b^2-c^2\right)^2}{4a^2}\overset{(**)}{\ge}(m_b-m_c)^2}$$

$$\text{Again, }\left(m_b^2-m_c^2\right)^2\overset{?}{\ge}\frac{9m_a^2}{4}(b-c)^2$$

$$\Leftrightarrow\left(\frac{\left(2c^2+2a^2-b^2\right)-\left(2a^2+2b^2-c^2\right)}{4}\right)^2\overset{?}{\ge}\frac{9\left(2b^2+2c^2-a^2\right)}{16}(b-c)^2$$

$$\Leftrightarrow 9\left(b^2-c^2\right)^2\overset{?}{\ge}9\left(2b^2+2c^2-a^2\right)(b-c)^2$$

$$\Leftrightarrow(b-c)^2\left((b+c)^2-\left((b+c)^2+(b-c)^2-a^2\right)\right)\overset{?}{\ge}0$$

$$\Leftrightarrow(b-c)^2\left(a^2-(b-c)^2\right)\overset{?}{\ge}0\Leftrightarrow(b-c)^2.4(s-b)(s-c)\overset{?}{\ge}0\rightarrow\text{true}$$

$$\therefore\boxed{\boxed{\left(m_b^2-m_c^2\right)^2\ge\frac{9m_a^2}{4}(b-c)^2}}\rightarrow(3)$$

Implementing (3) on a triangle with sides $\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$ whose medians as a consequence of elementary calculations $=\frac{a}{2},\frac{b}{2},\frac{c}{2}$ respectively, we arrive at :

$$\left(\frac{1}{4}\left(b^2-c^2\right)\right)^2\ge\frac{9.\frac{a^2}{4}}{4}.\left(\frac{2}{3}(m_b-m_c)\right)^2\Rightarrow\frac{\left(b^2-c^2\right)^2}{4a^2}\ge(m_b-m_c)^2$$

$$\Rightarrow(**)\Rightarrow(*)\text{ is true}\therefore m_b+m_c\ge\sqrt{h_a^2+\frac{9a^2}{4}}\ \forall\Delta ABC,''=''\text{ iff }b=c\ (QED)$$

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

In any ΔABC, we have

$$h_a=\frac{\sqrt{s(s-a)}.2\sqrt{(s-b)(s-c)}}{a}\overset{AM-GM}{\lesssim}\frac{\sqrt{s(s-a)}.\left((s-b)+(s-c)\right)}{a}=\frac{\sqrt{(b+c)^2-a^2}}{2}$$

$$\Rightarrow b+c\geq\sqrt{4{h_a}^2+a^2}.$$

Using this inequality to the triangle GBC, where G is the centroid of the triangle ABC, we obtain

$$GB+GC\geq\sqrt{4{h_G}^2+BC^2},$$

where h_G is the altitude from G. Since the area of ΔGBC is equal to $\frac{F}{3}$, then we have

$$h_G=\frac{h_a}{3}.$$

Also, we have $GB=\frac{2m_b}{3}, GC=\frac{2m_c}{3}$, then we get

$$m_b+m_c\geq\sqrt{{h_a}^2+\frac{9a^2}{4}}.$$

Equality holds iff $b=c$.