ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with I $\rightarrow$ incenter, the following relationship holds :

$$2\sqrt{\frac{AI+BI+CI}{2(AI+BI+CI+r)-4R}} \geq \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}$$

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$$2\sqrt{\frac{AI+BI+CI}{2(AI+BI+CI+r)-4R}} = \sqrt{\frac{4\sum_{cyc}\frac{r}{\sin\frac{A}{2}}}{2\sum_{cyc}\frac{r}{\sin\frac{A}{2}}-(4R-2r)}} =$$

$$\sqrt{\frac{\frac{2}{2R}\cdot\sum_{cyc}\frac{r}{\sin\frac{A}{2}}}{\frac{1}{2R}\sum_{cyc}\frac{r}{\sin\frac{A}{2}}-\frac{2R-r}{2R}}} = \sqrt{\frac{4\left(\prod_{cyc}\sin\frac{A}{2}\right).\frac{\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}}{\prod_{cyc}\sin\frac{A}{2}}}{2\left(\prod_{cyc}\sin\frac{A}{2}\right).\left(\frac{\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}}{\prod_{cyc}\sin\frac{A}{2}}\right)-\sum_{cyc}\sin^2\frac{A}{2}}}$$

$$= \sqrt{\frac{4\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}}{2\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}-\sum_{cyc}\sin^2\frac{A}{2}}} \overset{?}{\geq} \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}$$

$$\Leftrightarrow \frac{4\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}}{2\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}-\sum_{cyc}\sin^2\frac{A}{2}} - 2 \overset{?}{\geq} \frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}+\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}$$

$$\Leftrightarrow \frac{2\sum_{cyc}\sin^2\frac{A}{2}}{2\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}-\sum_{cyc}\sin^2\frac{A}{2}} \overset{?}{\geq} \frac{\sin^2\frac{B}{2}+\sin^2\frac{C}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\Leftrightarrow 2yz\sum_{cyc}x^2 \overset{?}{\geq} (y^2+z^2)\left(2\sum_{cyc}xy-\sum_{cyc}x^2\right)\left(x=\sin\frac{A}{2}, y=\sin\frac{B}{2}, z=\sin\frac{C}{2}\right)$$

$$\overset{\text{expanding and re-arranging}}{\Leftrightarrow} x^2(y+z)^2+(y^2+z^2)^2 \overset{?}{\geq} 2xyz(y+z)+2x(y^3+z^3)$$

$$\Leftrightarrow x^2(y+z)^2+(y^2+z^2)^2 \overset{?}{\geq} 2xyz(y+z)+2x(y+z)(y^2+z^2-yz)$$

$$= 2x(y+z)(yz+y^2+z^2-yz) \Leftrightarrow x^2(y+z)^2+(y^2+z^2)^2 \overset{?}{\geq} 2x(y+z)(y^2+z^2)$$

$$\Leftrightarrow \left(x(y+z)-(y^2+z^2)\right)^2 \overset{?}{\geq} 0 \rightarrow \text{true}$$

$$\therefore 2\sqrt{\frac{AI+BI+CI}{2(AI+BI+CI+r)-4R}} \geq \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}\ \forall\,\Delta\text{ ABC (QED)}$$