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In any Δ ABC, the following relationship holds :

$$2\sum_{cyc}\frac{m_a}{h_a}\geq\frac{2s}{\sqrt[3]{abc}}+\frac{1}{\sqrt[3]{h_ah_bh_c}}\sum_{cyc}h_a$$

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$$2\sum_{cyc}\frac{m_a}{h_a}-\frac{1}{\sqrt[3]{h_ah_bh_c}}\sum_{cyc}h_a\overset{\text{Tereshin}}{\geq}2\sum_{cyc}\frac{b^2+c^2}{4R.\frac{bc}{2R}}-\frac{1}{\sqrt[3]{\frac{2r^2s^2}{R}}}\cdot\frac{s^2+4Rr+r^2}{2R}$$

$$\overset{\text{Gerretsen}}{\geq}\frac{\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc}{abc}-\frac{1}{\sqrt[3]{\frac{r^2(27Rr+5r(R-2r))}{R}}}\cdot\frac{s^2+4Rr+r^2}{2R}$$

$$\overset{\text{Euler}}{\geq}\frac{2s(s^2+4Rr+r^2)-12Rrs}{4Rrs}-\frac{1}{\sqrt[3]{\frac{r^2(27Rr)}{R}}}\cdot\frac{s^2+4Rr+r^2}{2R}$$

$$=\frac{s^2-2Rr+r^2}{2Rr}-\frac{s^2+4Rr+r^2}{6Rr}=\frac{s^2-5Rr+r^2}{3Rr}\overset{?}{\geq}\frac{2s}{\sqrt[3]{abc}}=\frac{2s}{\sqrt[3]{4Rrs}}$$

$$\Leftrightarrow\frac{(s^2-5Rr+r^2)^3}{27R^3r^3}\overset{?}{\geq}\frac{8s^3}{4Rrs}\Leftrightarrow(s^2-5Rr+r^2)^3-54R^2r^2s^2\boxed{\overset{?}{\underset{(*)}{\geq}}}0$$

and $\because (s^2-16Rr+5r^2)^3\overset{\text{Gerretsen}}{\geq}0$ $\therefore$ in order to prove $(*)$, it suffices to prove :

LHS of $(*)\geq(s^2-16Rr+5r^2)^3\Leftrightarrow(33R-12r)s^4-rs^2(747R^2-450Rr+72r^2)$

$$+r^2(3971R^3-3765R^2r+1185Rr^2-124r^3)\boxed{\underset{\geq}{(**)}}0\text{ and}$$

$\because (33R-12r)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0$ $\therefore$ in order to prove $(**)$, it suffices to prove : LHS of $(**)\geq(33R-12r)(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow(309R^2-264Rr+48r^2)s^2\boxed{\underset{\geq}{(***)}}r(4477R^3-4587R^2r+1560Rr^2-176r^3)$$

Now, LHS of $(***)\overset{\text{Gerretsen}}{\geq}(309R^2-264Rr+48r^2)(16Rr-5r^2)\overset{?}{\geq}$

$r(4477R^3-4587R^2r+1560Rr^2-176r^3)\Leftrightarrow 467t^3-1182t^2+528t-64\overset{?}{\geq}0$

$\left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(343t^2+124t(t-2)+32)\overset{?}{\geq}0\rightarrow$ true $\because t\overset{\text{Euler}}{\geq}2$

$\Rightarrow(***)\Rightarrow(**)\Rightarrow(*)$ is true $\therefore 2\sum_{cyc}\frac{m_a}{h_a}-\frac{1}{\sqrt[3]{h_ah_bh_c}}\sum_{cyc}h_a\geq\frac{2s}{\sqrt[3]{abc}}\Rightarrow$

$2\sum_{cyc}\frac{m_a}{h_a}\geq\frac{2s}{\sqrt[3]{abc}}+\frac{1}{\sqrt[3]{h_ah_bh_c}}\sum_{cyc}h_a\ \forall\ \Delta$ ABC, " = " iff Δ ABC is equilateral (QED)