ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\sqrt{\frac{2m_a}{h_a}\left(\frac{r_b}{r_c}+\frac{r_c}{r_b}\right)}\ge\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}$$

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$$\text{Let } a=y+z, b=z+x \text{ and } c=x+y \therefore 2s=a+b+c=2(x+y+z)$$

$$\Rightarrow s=x+y+z\therefore s-a=x, s-b=y\therefore\frac{r_a}{r_b}+\frac{r_b}{r_a}\ge\frac{a}{b}+\frac{b}{a}$$

$$\Leftrightarrow\frac{y}{x}+\frac{x}{y}\ge\frac{y+z}{z+x}+\frac{z+x}{y+z}\Leftrightarrow(x^2+y^2)(z+x)(y+z)\ge xy\left((z+x)^2+(y+z)^2\right)$$

$$\Leftrightarrow z(x^3+y^3)-xyz(x+y)+z^2(x-y)^2\ge 0\Leftrightarrow z(x+y)(x-y)^2+z^2(x-y)^2$$

$$\ge 0\to\text{true}\therefore\frac{r_a}{r_b}+\frac{r_b}{r_a}\ge\frac{a}{b}+\frac{b}{a}\text{ and analogs}\Rightarrow\frac{r_b}{r_c}+\frac{r_c}{r_b}\ge\frac{b}{c}+\frac{c}{b}$$

$$\therefore\sqrt{\frac{2m_a}{h_a}\left(\frac{r_b}{r_c}+\frac{r_c}{r_b}\right)}\ge\sqrt{\frac{2m_a}{h_a}\left(\frac{b}{c}+\frac{c}{b}\right)}\overset{\text{Tereshin}}{\ge}\sqrt{\frac{2\left(\frac{b^2+c^2}{4R}\right)}{\frac{bc}{2R}}\left(\frac{b}{c}+\frac{c}{b}\right)}\Rightarrow\sqrt{\frac{2m_a}{h_a}\left(\frac{r_b}{r_c}+\frac{r_c}{r_b}\right)}$$

$$\ge\frac{b}{c}+\frac{c}{b}\text{ and analogs}\therefore\sum_{cyc}\sqrt{\frac{2m_a}{h_a}\left(\frac{r_b}{r_c}+\frac{r_c}{r_b}\right)}\ge\sum_{cyc}\left(\frac{b}{c}+\frac{c}{b}\right)=\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}$$

$$\forall\,\Delta\,\text{ABC},''=''\text{ iff }\Delta\,\text{ABC is equilateral (QED)}$$