ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{n_a^2}{h_a^2} \le \frac{m_a}{h_a} \cdot \sqrt{\frac{R}{2r} + \frac{r_a - r}{r_a} \cdot \left(\frac{r_b - r_c}{2(s-a)}\right)^2}$$

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$$\text{Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2+c^2) - 2sbc$$

$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 +$$

$$s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)}{bc}$$

$$= as^2 - s(a^2 - (b-c)^2) = sa(s-a) + s(b-c)^2$$

$$\therefore an_a^2 = sa(s-a) + s(b-c)^2 \to (1)$$

$$\text{Now, } \frac{m_a}{h_a} \cdot \sqrt{\frac{R}{2r} + \frac{r_a - r}{r_a} \cdot \left(\frac{r_b - r_c}{2(s-a)}\right)^2} \overset{\text{Panaitopol}}{\ge} \frac{m_a^2}{h_a^2} + \frac{\frac{rs}{s-a} - \frac{rs}{s}}{\frac{rs}{s-a}} \cdot \left(\frac{\frac{rs}{s-b} - \frac{rs}{s-c}}{2(s-a)}\right)^2$$

$$= \frac{m_a^2}{h_a^2} + \frac{\frac{rsa}{s(s-a)}}{\frac{rs}{s-a}} \cdot \frac{r^2s^2(b-c)^2}{4r^4s^2} = \frac{a^2m_a^2}{4s^2r^2} + \frac{as(b-c)^2}{4s^2r^2} \overset{\text{via (1)}}{=}$$

$$\frac{a^2m_a^2}{4s^2r^2} + \frac{a\left(an_a^2 - sa(s-a)\right)}{4s^2r^2} = \frac{n_a^2}{\frac{4s^2r^2}{a^2}} + \frac{a^2}{4s^2r^2} \cdot \left(m_a^2 - s(s-a)\right)$$

$$= \frac{n_a^2}{h_a^2} + \frac{a^2}{4s^2r^2} \cdot \left(m_a^2 - s(s-a)\right) \ge \frac{n_a^2}{h_a^2}$$

$$\left(\because m_a \overset{\text{Lascu}}{\ge} \frac{b+c}{2}\cos\frac{A}{2} \overset{\text{A-G}}{\ge} \sqrt{bc} \cdot \frac{\sqrt{s(s-a)}}{\sqrt{bc}} \Rightarrow m_a^2 - s(s-a) \ge 0\right)$$

$$\therefore \frac{n_a^2}{h_a^2} \le \frac{m_a}{h_a} \cdot \sqrt{\frac{R}{2r} + \frac{r_a - r}{r_a} \cdot \left(\frac{r_b - r_c}{2(s-a)}\right)^2} \quad \forall \Delta \text{ ABC},$$

$$'' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$