ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC, the following relationship holds :

$$m_a \geq max\left(\sqrt[3]{p_a r_b r_c}, \sqrt[3]{\frac{p_a h_a (r_b + r_c)}{2}}\right)$$

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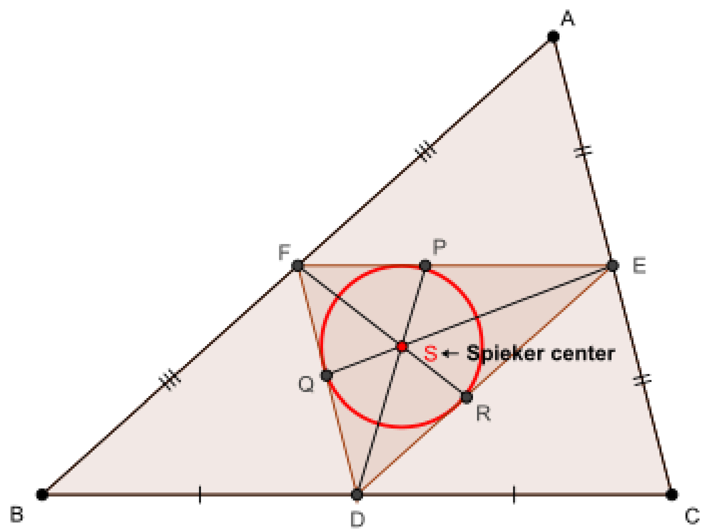
Solution 1 by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC *at* X *and* $m(\measuredangle BAX) = \alpha$ *and* $m(\measuredangle CAX) = \beta$ (*say*) *and* inradius of Δ DEF $= r'$(*say*)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta \text{ DEF}, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) *and* using cosine l*a*w on Δ AFS *and* Δ AES, we *a*rrive *at* : $AS^2 =$

$$\frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} = \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}}$$

$$+\frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Again,}\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$
$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$
$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$
$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$
$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$
$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$
$$\text{Also,}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$
$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$
$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$
$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$
$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$
$$\text{Via sine law on }\Delta\,AFS,\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$
$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,AES,b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$
$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta$$
$$=rs\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$
$$\Rightarrow p_a^2-m_a^2=\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)-m_a^2$$
$$=\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc\right)-\left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$=\frac{4(a+b+c)\left(b^3+c^3-abc\right)-\left(2b^2+2c^2-a^2\right)(b+c)^2}{4(2s+a)^2}$$

$$=\frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2\left(b^2-c^2\right)^2}{4(2s+a)^2}$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\left(\left(a^2+2a(b+c)+(b+c)^2\right)+\left((b+c)^2+2a(b+c)+a^2\right)-a^2\right)$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\left(2(a+b+c)^2-a^2\right)=\frac{(b-c)^2\left(8s^2-a^2\right)}{4(2s+a)^2}$$

$$\therefore p_a^2-m_a^2\overset{(\bullet)}{=}\frac{(b-c)^2\left(8s^2-a^2\right)}{4(2s+a)^2}$$

$$\text{Now, } m_a^3\overset{?}{\geq}p_a r_b r_c\Leftrightarrow m_a^6\overset{?}{\geq}p_a^2 s^2(s-a)^2\Leftrightarrow\frac{m_a^4}{s^2(s-a)^2}-1\overset{?}{\geq}\frac{p_a^2}{m_a^2}-1$$

$$\Leftrightarrow\frac{\left(m_a^2+s(s-a)\right)\left(m_a^2-s(s-a)\right)}{s^2(s-a)^2}\overset{?}{\geq}\frac{p_a^2-m_a^2}{m_a^2}$$

$$\overset{\text{via }(\bullet)}{\Leftrightarrow}\frac{\left(m_a^2+s(s-a)\right)\left(s(s-a)+\frac{(b-c)^2}{4}-s(s-a)\right)}{s^2(s-a)^2}\overset{?}{\geq}\frac{\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}}{m_a^2}$$

$$\Leftrightarrow\frac{\left(m_a^2+s(s-a)\right).\frac{(b-c)^2}{4}}{s^2(s-a)^2}\underset{(\blacksquare)}{\overset{?}{\geq}}\frac{\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}}{m_a^2}\text{ and }\because(b-c)^2\geq 0\therefore\text{ in order}$$

$$\text{to prove }(\blacksquare)\text{, it suffices to prove : }\frac{m_a^2\left(m_a^2+s(s-a)\right)}{s^2(s-a)^2}\overset{(\blacksquare\blacksquare)}{>}\frac{8s^2-a^2}{(2s+a)^2}$$

$$\text{But, LHS of }(\blacksquare\blacksquare)\overset{\text{Lascu + A-G}}{\geq}\frac{s(s-a)\left(s(s-a)+s(s-a)\right)}{s^2(s-a)^2}=2\overset{?}{>}\frac{8s^2-a^2}{(2s+a)^2}$$

$$\Leftrightarrow 8s^2+8sa+2a^2\overset{?}{>}8s^2-a^2\Leftrightarrow 8sa+3a^2\overset{?}{>}0\to\text{true}\Rightarrow(\blacksquare\blacksquare)\Rightarrow(\blacksquare)\text{ is true}$$

$$\therefore m_a^3\geq p_a r_b r_c\Rightarrow m_a\geq\sqrt[3]{p_a r_b r_c}\to(\square)\therefore m_a^3\geq p_a r_b r_c\overset{?}{\geq}\frac{p_a h_a(r_b+r_c)}{2}$$

$$\Leftrightarrow 2s(s-a)\overset{?}{\geq}\frac{bc}{2R}.4R\cos^2\frac{A}{2}=2bc.\frac{s(s-a)}{bc}\Leftrightarrow 2s(s-a)\overset{?}{\geq}2s(s-a)\to\text{true}$$

$$\therefore m_a^3\geq\frac{p_a h_a(r_b+r_c)}{2}\Rightarrow m_a\geq\sqrt[3]{\frac{p_a h_a(r_b+r_c)}{2}}\to(\square\square)\therefore(\square),(\square\square)$$

$$\Rightarrow m_a\geq\sqrt[3]{p_a r_b r_c},\sqrt[3]{\frac{p_a h_a(r_b+r_c)}{2}}$$

$$\Rightarrow m_a\geq max\left(\sqrt[3]{p_a r_b r_c},\sqrt[3]{\frac{p_a h_a(r_b+r_c)}{2}}\right),''=''\text{ iff }b=c\text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have the following known formulas (see [1, pp. 1]),

$$p_a{}^2 = s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}. \qquad (1)$$

$$m_a{}^2 = s(s-a) + \frac{(b-c)^2}{4}, \qquad (2)$$

By AM − GM inequality, we have

$$\sqrt[3]{p_a r_b r_c}^2 \le \frac{p_a{}^2 + r_b r_c + r_b r_c}{3} \overset{(1)}{\cong} s(s-a) + \frac{s(3s+a)(b-c)^2}{3(2s+a)^2}$$

$$= s(s-a) + \left(\frac{1}{4} - \frac{a(8s+3a)}{12(2s+a)^2}\right)(b-c)^2 \le s(s-a) + \frac{(b-c)^2}{4} \overset{(2)}{\cong} m_a{}^2,$$

then

$$m_a \ge \sqrt[3]{p_a r_b r_c}.$$

Also, we have

$$\frac{h_a(r_b+r_c)}{2} = \frac{r_b r_c h_a}{2}\left(\frac{1}{r_b} + \frac{1}{r_c}\right) = r_b r_c . \frac{F}{a} . \frac{(s-b)+(s-c)}{F} = r_b r_c .$$

Therefore

$$m_a \ge \sqrt[3]{p_a r_b r_c} = max\left(\sqrt[3]{p_a r_b r_c}, \sqrt[3]{\frac{p_a h_a(r_b+r_c)}{2}}\right).$$

Equality holds if and only if $b = c$.

[1] Bogdan Fuștei, Mohamed Amine Ben Ajiba,
SPIEKER'S CEVIANS IN THE GEOMETRY OF TRIANGLE-www.ssmrmh.ro