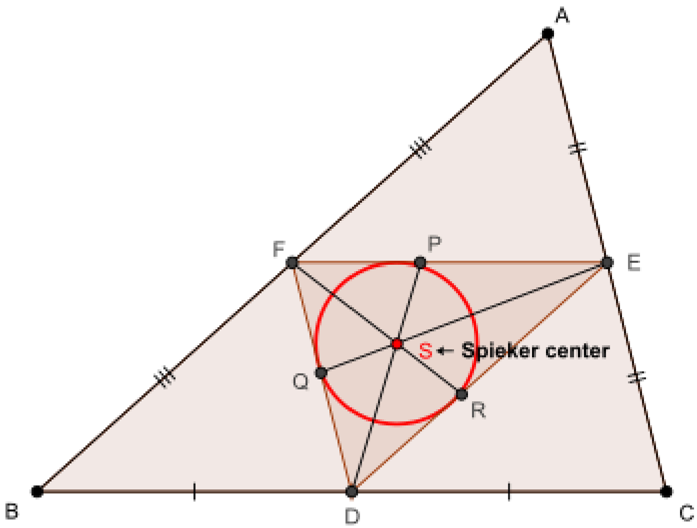
ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$R\sum_{cyc} h_a \geq \frac{4}{3}\sin\omega \sum_{cyc} p_a w_a$$

Proposed by Bogdan Fuștei-Romania
Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$(say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$$\because \text{ Spieker center is incenter of } \Delta \text{ DEF}, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \rightarrow (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at : $AS^2 =$

$$\frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} = \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \stackrel{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}}$$

$$+\frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Again,}\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\stackrel{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Also,}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\stackrel{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\stackrel{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS,}\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\stackrel{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta\stackrel{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta$$

$$=rs\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2-m_a^2=\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)-m_a^2$$

$$=\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc\right)-\left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$= \frac{4(a+b+c)(b^3+c^3-abc)-(2b^2+2c^2-a^2)(b+c)^2}{4(2s+a)^2}$$

$$= \frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2(b^2-c^2)^2}{4(2s+a)^2}$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\left((a^2+2a(b+c)+(b+c)^2)+((b+c)^2+2a(b+c)+a^2)-a^2\right)$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\left(2(a+b+c)^2-a^2\right) = \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2 - m_a^2 \overset{(\bullet)}{=} \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\text{Now, } p_a^2 w_a^2 \overset{?}{\leq} m_a^4 \Leftrightarrow \frac{p_a^2}{m_a^2} - 1 \overset{?}{\leq} \frac{m_a^2}{w_a^2} - 1 \overset{\text{via } (\bullet)}{\Leftrightarrow}$$

$$\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2 m_a^2} \underset{(\blacksquare)}{\overset{?}{\leq}} \frac{s(s-a)+\frac{(b-c)^2}{4}-\left(s(s-a)-\frac{s(s-a)(b-c)^2}{(b+c)^2}\right)}{w_a^2}$$

$$= \frac{(b-c)^2(4s(s-a)+(b+c)^2)}{4(b+c)^2 w_a^2} \text{ and } \because (b-c)^2 \geq 0 \text{ and } w_a^2 \leq m_a^2$$

$$\therefore \text{in order to prove } (\blacksquare), \text{it suffices to prove : } \frac{8s^2-a^2}{(2s+a)^2} < \frac{4s(s-a)+(b+c)^2}{(b+c)^2}$$

$$\Leftrightarrow \frac{8s^2-a^2}{(2s+a)^2} < \frac{4s(s-a)+(2s-a)^2}{(2s-a)^2} \Leftrightarrow \frac{8s^2-a^2}{(2s+a)^2} < \frac{8s^2-8sa+a^2}{(2s-a)^2}$$

$$\Leftrightarrow (8s^2-8sa+a^2)(2s+a)^2 > (8s^2-a^2)(2s-a)^2$$

$$\Leftrightarrow 16s^3-12s^2a-4sa^2+a^3 > 0 \Leftrightarrow 12s^2(s-a)+4s(s-a)(s+a)+a^3 > 0$$

$$\rightarrow \text{true} \because s-a > 0 \therefore p_a^2 w_a^2 \leq m_a^4 \Rightarrow p_a w_a \leq m_a^2 \text{ and analogs}$$

$$\Rightarrow \frac{4}{3}\sin\omega . \sum_{\text{cyc}} p_a w_a \leq \frac{4}{3} . \frac{2rs}{\sqrt{\sum_{\text{cyc}} a^2b^2}} . \sum_{\text{cyc}} m_a^2$$

$$= \frac{4}{3} . \frac{2rs}{\sqrt{\sum_{\text{cyc}} a^2b^2}} . \frac{3}{4} . 2(s^2-4Rr-r^2) \overset{?}{\leq} R. \sum_{\text{cyc}} h_a = \frac{\sum_{\text{cyc}} ab}{2}$$

$$\Leftrightarrow \frac{1}{4}\left(\sum_{\text{cyc}} ab\right)^2 \overset{?}{\geq} \frac{16r^2s^2(s^2-4Rr-r^2)^2}{\sum_{\text{cyc}} a^2b^2}$$

$$\Leftrightarrow \left(\sum_{\text{cyc}} a^2b^2\right)\left(\sum_{\text{cyc}} ab\right)^2 \underset{(?)}{\overset{?}{\geq}} 64r^2s^2(s^2-4Rr-r^2)^2$$

$$\text{Now, } \sum_{\text{cyc}} a^2b^2 \geq \frac{1}{3}\left(\sum_{\text{cyc}} ab\right)^2 \geq 8Rrs^2 \Rightarrow \text{LHS of } (?) \geq$$

$$8Rrs^2(s^2+4Rr+r^2)^2 \overset{?}{\geq} 64r^2s^2(s^2-4Rr-r^2)^2$$

$$\Leftrightarrow (R-2r)s^4 - 6rs^4 + rs^2(8R^2+66Rr+16r^2)$$

$$+r^2(16R^3-120R^2r-63Rr^2-8r^3) \underset{(??)}{\overset{?}{\geq}} 0$$

$$\text{Again, LHS of } (\square\square) \overset{\text{Gerretsen}}{\geq} \begin{pmatrix} (R-2r)(16Rr-5r^2)-6r(4R^2+4Rr+3r^2) \\ +r(8R^2+66Rr+16r^2) \end{pmatrix} s^2$$

$$+r^2(16R^3-120R^2r-63Rr^2-8r^3) = r^2(5R+8r)s^2$$

$$+r^2(16R^3-120R^2r-63Rr^2-8r^3) \overset{\text{Gerretsen}}{\geq} r^2(5R+8r)(16Rr-5r^2)$$

$$+r^2(16R^3-120R^2r-63Rr^2-8r^3) \overset{?}{\geq} 0 \Leftrightarrow 2t^3-5t^2+5t-6 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(2t^2-t+3) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\square\square) \Rightarrow (\square) \text{ is true} \Rightarrow$$

$$R.\sum_{cyc} h_a \geq \frac{4}{3}\sin\omega.\sum_{cyc} p_a w_a \ \forall \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$