

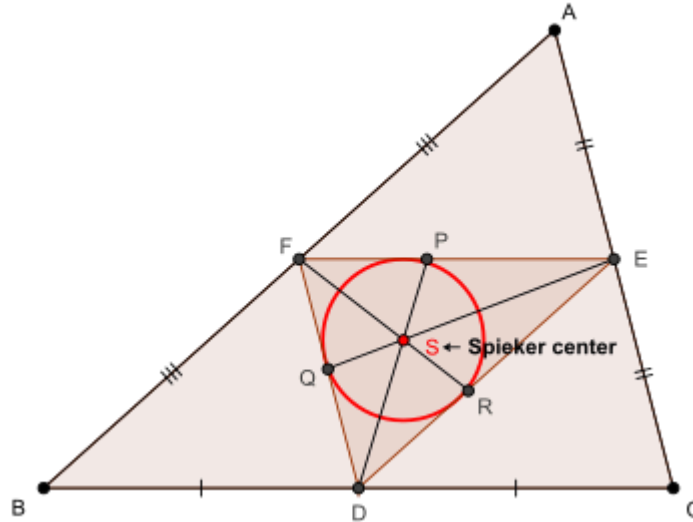
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In any ΔABC , the following relationship holds :

$$\sum_{\text{cyc}} \frac{|b-c|}{w_a} \geq 2 \sum_{\text{cyc}} \sqrt{\frac{p_a}{m_a} - 1}$$

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Let AS produced meet BC at X and $m(\angle BAX) = \alpha$ and $m(\angle CAX) = \beta$ (say)
and inradius of $\Delta DEF = r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2 \sum \left(\frac{a^2}{4}\right) \left(\frac{b^2}{4}\right) - \sum \frac{a^4}{16} = \frac{1}{16} \left(2 \sum a^2 b^2 - \sum a^4\right) = \frac{16r^2 s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r' \left(\frac{\frac{a}{2} + \frac{b}{2} + \frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

\therefore Spieker center is incenter of ΔDEF , $\therefore m(\angle AFS) = B + \frac{C}{2} = \frac{2B + C}{2} = \frac{B + \pi - A}{2}$

$$= \frac{\pi}{2} - \frac{A - B}{2} \text{ and } m(\angle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A - C}{2} \rightarrow (2)$$

Via (1), (2) and using cosine law on ΔAFS and ΔAES , we arrive at : $AS^2 =$
 $\frac{r^2}{4 \sin^2 \frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2 \sin \frac{C}{2}}\right) \left(\frac{c}{2}\right) \sin \frac{A - B}{2} = \frac{r^2}{4 \sin^2 \frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2 \sin \frac{B}{2}}\right) \left(\frac{b}{2}\right) \sin \frac{A - C}{2}$

$$\Rightarrow 2AS^2 \stackrel{(i)}{=} \frac{r^2}{4 \sin^2 \frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2 \sin \frac{C}{2}}\right) \left(\frac{c}{2}\right) \sin \frac{A - B}{2} + \frac{r^2}{4 \sin^2 \frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2 \sin \frac{B}{2}}\right) \left(\frac{b}{2}\right) \sin \frac{A - C}{2}$$

$$\text{Again, } \left(\frac{2r}{2 \sin \frac{C}{2}}\right) \left(\frac{c}{2}\right) \sin \frac{A - B}{2} + \left(\frac{2r}{2 \sin \frac{B}{2}}\right) \left(\frac{b}{2}\right) \sin \frac{A - C}{2}$$

$$\begin{aligned}
 &= \frac{r}{2} \left(4R \cos \frac{C}{2} \sin \frac{A-B}{2} + 4R \cos \frac{B}{2} \sin \frac{A-C}{2} \right) \\
 &= Rr \left(2 \sin \frac{A+B}{2} \sin \frac{A-B}{2} + 2 \sin \frac{A+C}{2} \sin \frac{A-C}{2} \right) \\
 &= Rr \left(1 - 2 \sin^2 \frac{B}{2} + 1 - 2 \sin^2 \frac{C}{2} - 2 \left(1 - 2 \sin^2 \frac{A}{2} \right) \right) \\
 &= 2Rr \left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc} \right) \\
 &= \frac{Rr}{8Rrs} (2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2) \\
 &= \frac{4(b+c)bc \sin^2 \frac{A}{2} - 2a \cdot 2bc \cos A}{8s} = \frac{bc \left((2s-a) \sin^2 \frac{A}{2} - a \left(1 - 2 \sin^2 \frac{A}{2} \right) \right)}{2s} \\
 &= \frac{bc \left((2s+a) \sin^2 \frac{A}{2} - a \right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr \\
 &\Rightarrow - \left(\frac{2r}{2 \sin \frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin \frac{A-B}{2} - \left(\frac{2r}{2 \sin \frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin \frac{A-C}{2} \\
 &\quad \stackrel{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr
 \end{aligned}$$

$$\begin{aligned}
 \text{Also, } \frac{r^2}{4 \sin^2 \frac{B}{2}} + \frac{r^2}{4 \sin^2 \frac{C}{2}} &= \frac{r^2}{4} \left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)} \right) \\
 &= \frac{r^2}{4r^2s} (ca(s-b) + ab(s-c)) = \frac{ab+ca}{4} - 2Rr \stackrel{(**)}{=} \frac{r^2}{4 \sin^2 \frac{B}{2}} + \frac{r^2}{4 \sin^2 \frac{C}{2}} \\
 \text{(i), (*), (**)} \Rightarrow 2AS^2 &= \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s} \\
 &= \frac{(a+b+c)(b^2+c^2+ab+ca) - (2a+b+c)(c+a-b)(a+b-c)}{4s} \\
 &= \frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s} \stackrel{(ii)}{\Rightarrow} 2AS^2 = \frac{b^3+c^3-abc+a(4m_a^2)}{4s}
 \end{aligned}$$

$$\begin{aligned}
 \text{Via sine law on } \triangle AFS, \frac{r}{2 \sin \frac{C}{2} \sin \alpha} &= \frac{AS}{\cos \frac{A-B}{2}} = \frac{4s}{(a+b) \sin \frac{C}{2}} \\
 \Rightarrow c \sin \alpha &\stackrel{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \triangle AES, b \sin \beta \stackrel{(***)}{=} \frac{r(a+c)}{2AS}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, } [BAX] + [BAX] &= [ABC] \Rightarrow \frac{1}{2} p_a c \sin \alpha + \frac{1}{2} p_a b \sin \beta \\
 = rs &\stackrel{\text{via (***) and (***)}}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a} AS
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow p_a^2 - m_a^2 &= \frac{2s}{(2s+a)^2} (b^3+c^3-abc+a(4m_a^2)) - m_a^2 \\
 &= \frac{2s}{(2s+a)^2} (b^3+c^3-abc) - \left(1 - \frac{8sa}{(2s+a)^2} \right) m_a^2 \\
 &= \frac{4(a+b+c)(b^3+c^3-abc) - (2b^2+2c^2-a^2)(b+c)^2}{4(2s+a)^2} \\
 &= \frac{a^2(b-c)^2 + 4a(b+c)(b-c)^2 + 2(b^2-c^2)^2}{4(2s+a)^2}
 \end{aligned}$$

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$$\begin{aligned}
 &= \frac{(b-c)^2}{4(2s+a)^2} \left((a^2 + 2a(b+c) + (b+c)^2) + ((b+c)^2 + 2a(b+c) + a^2) - a^2 \right) \\
 &= \frac{(b-c)^2}{4(2s+a)^2} (2(a+b+c)^2 - a^2) = \frac{(b-c)^2(8s^2 - a^2)}{4(2s+a)^2} \\
 &\quad \therefore p_a^2 - m_a^2 \stackrel{(*)}{=} \frac{(b-c)^2(8s^2 - a^2)}{4(2s+a)^2}
 \end{aligned}$$

Now, $\frac{(b-c)^2}{w_a^2} \stackrel{?}{\geq} 4 \left(\frac{p_a}{m_a} - 1 \right) = \frac{4(p_a^2 - m_a^2)}{m_a(p_a + m_a)} \stackrel{\text{via } (*)}{=} \frac{(b-c)^2(8s^2 - a^2)}{(2s+a)^2 m_a(p_a + m_a)}$ and $\therefore (b-c)^2 \geq 0$ and $m_a(p_a + m_a) \geq 2m_a^2 \geq 2w_a^2$

$\left(\therefore p_a^2 - m_a^2 = \frac{(b-c)^2(8s^2 - a^2)}{4(2s+a)^2} \geq 0 \therefore p_a \geq m_a \right) \therefore$ in order to prove (\blacksquare) ,

it suffices to prove : $1 > \frac{8s^2 - a^2}{2(2s+a)^2} \Leftrightarrow 8sa + 3a^2 > 0 \rightarrow \text{true} \Rightarrow (\blacksquare)$ is true

$$\therefore \frac{(b-c)^2}{w_a^2} \geq 4 \left(\frac{p_a}{m_a} - 1 \right) \Rightarrow \frac{|b-c|}{w_a} \geq 2 \sqrt{\frac{p_a}{m_a} - 1} \text{ and analogs}$$

$$\Rightarrow \sum_{\text{cyc}} \frac{|b-c|}{w_a} \geq 2 \sum_{\text{cyc}} \sqrt{\frac{p_a}{m_a} - 1} \forall \Delta ABC, " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$