ROMANIAN MATHEMATICAL MAGAZINE

If $a=\min\{a,b,c\}$, then in acute Δ ABC, the following relationship holds :

$$2\left(\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)\ge\frac{R}{r}+\frac{h_b+h_c}{h_a}$$

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$$\frac{b+c}{a}\overset{?}{\ge}\frac{R}{r}=\frac{abcs}{4F^2}=\frac{2abc}{(b+c-a)(c+a-b)(a+b-c)}$$

$$\Leftrightarrow (b+c)(a+b-c).(b+c-a)(c+a-b)\overset{?}{\ge}2a^2bc$$

$$\Leftrightarrow (b+c)(a+b-c).(b+c-a)(c+a-b)\overset{?}{\ge}2a^2bc$$

$$\Leftrightarrow \left(ab+b^2-bc+ca+bc-c^2\right)\left(bc+ab-b^2+c^2+ca-bc-ca-a^2+ab\right)$$
$$\overset{?}{\ge}2a^2bc$$

$$\Leftrightarrow 2a^2b^2+2a^2bc+2ab\left(b^2-c^2\right)-\left(a^2+b^2-c^2\right)\left(ab+ac+b^2-c^2\right)\overset{?}{\ge}2a^2bc$$

$$\Leftrightarrow 2a^2b^2-\left(a^2+b^2-c^2\right)(ab+ac)+2ab\left(b^2-c^2\right)-2ab\left(b^2-c^2\right).\cos C\overset{?}{\ge}0$$

$$\Leftrightarrow 2a^2b^2-a^2(ab+ac)-\left(b^2-c^2\right)(ab+ac)+2ab\left(b^2-c^2\right).2\sin^2\frac{C}{2}\overset{?}{\ge}0$$

$$\Leftrightarrow a^2\left(2b^2-ab-ac\right)-\left(b^2-c^2\right)(ab+ac)+\left(b^2-c^2\right)\left(c^2-(a-b)^2\right)\overset{?}{\ge}0$$

$$\Leftrightarrow a^2\left(2b^2-ab-ac\right)+\left(b^2-c^2\right)\left(c^2-a^2-b^2+2ab-ab-ac\right)\overset{?}{\ge}0$$

$$\Leftrightarrow \left(\left(a^2-b^2+c^2\right)+\left(b^2-c^2\right)\right)\left(2b^2-ab-ac\right)$$
$$+\left(b^2-c^2\right)\left(c^2-a^2-b^2+ab-ac\right)\overset{?}{\ge}0$$

$$\Leftrightarrow \left(c^2+a^2-b^2\right)\left(2b^2-ab-ac\right)+\left(b^2-c^2\right)\left(b^2+c^2-a^2-2ac\right)\overset{?}{\ge}0$$

$$\Leftrightarrow \left(b^2-c^2\right)\left(b^2+c^2-2ac\right)-\left(b^2-c^2\right)\left(2b^2-ab-ac\right)+a^2\left(2b^2-ab-ac\right)$$
$$-a^2\left(b^2-c^2\right)\overset{?}{\ge}0$$

$$\Leftrightarrow \left(b^2-c^2\right)\left(\left(c^2-ca\right)-\left(b^2-ab\right)\right)+a^2\left(\left(c^2-ca\right)+\left(b^2-ab\right)\right)\overset{?}{\ge}0$$

$$\Leftrightarrow \left(c^2-ca\right)\left(b^2-c^2+a^2\right)+\left(b^2-ab\right)\left(a^2+c^2-b^2\right)\overset{?}{\ge}0$$

$$\Leftrightarrow c(c-a)\left(a^2+b^2-c^2\right)+b(b-a)\left(c^2+a^2-b^2\right)\overset{?}{\ge}0\to\text{true}$$

$$\because \Delta\,ABC\text{ being acute}\Rightarrow\left(a^2+b^2-c^2\right),\left(c^2+a^2-b^2\right)>0\text{ and }a=\min\{a,b,c\}$$

$$\Rightarrow(c-a),(b-a)\ge0\therefore\frac{R}{r}+\frac{h_b+h_c}{h_a}\le\frac{b+c}{a}+\frac{ca+ab}{bc}=\frac{b}{a}+\frac{a}{b}+\frac{c}{a}+\frac{a}{c}$$

$$=\frac{c^2+a^2}{ca}+\frac{a^2+b^2}{ab}\overset{\text{Tereshin}}{\le}\frac{4Rm_b}{2Rh_b}+\frac{4Rm_c}{2Rh_c}=2\left(\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)$$

$$\therefore 2\left(\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)\ge\frac{R}{r}+\frac{h_b+h_c}{h_a}\text{ in acute }\Delta\,ABC\text{ with }a=\min\{a,b,c\}\text{ (QED)}$$