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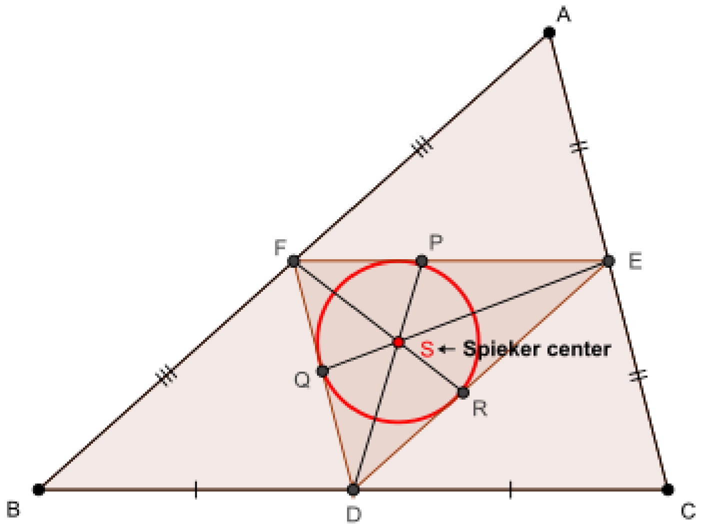
In any Δ ABC with
$p_a \rightarrow$ Spieker cevian, the following relationship holds :

$$\sum_{cyc} \frac{4p_a - h_a - r_a}{m_a} \le \frac{m_a + m_b + m_c}{r}$$

Proposed by Bogdan Fuștei-Romania

Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$$\because \text{ Spieker center is incenter of } \Delta \text{ DEF}, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \rightarrow (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now,}\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again,}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS},\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES}, b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}.\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$

$$\therefore\boxed{p_a^2\overset{(\blacksquare)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)}$$

$$\text{Now, }b^3+c^3-abc+a\left(4m_a^2\right)=b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)$$

$$= (b+c)\left(b^2-bc+c^2\right)+a\left(b^2-bc+c^2\right)+a\left(b^2+c^2-a^2\right)$$
$$= 2s\left(b^2-bc+c^2\right)+a\left(b^2-bc+c^2+bc-a^2\right)$$
$$= (2s+a)\left(b^2-bc+c^2\right)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$
$$= (2s+a)\left(b^2-bc+c^2\right)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a)\left(b^2-bc+c^2\right)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a).$$
$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$
$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$
$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$
$$\therefore \boxed{b^3+c^3-abc+a\left(4m_a^2\right)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$
$$\therefore (\bullet),(\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$
$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$
$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$
$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$
$$\Rightarrow p_a^2\overset{(\bullet\bullet\bullet)}{=}s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$
$$\text{Now, } p_a\le\frac{(b+c)^2}{16r}\Leftrightarrow p_a.\frac{2rs}{a}\le\frac{(b+c)^2}{8}.\frac{s}{a}\Leftrightarrow p_ah_a\le\frac{s(b+c)^2}{8a}\Leftrightarrow$$
$$p_a^2h_a^2\le\frac{s^2(b+c)^4}{64a^2}\overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$
$$\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)\left(s(s-a)-\frac{s(s-a)(b-c)^2}{a^2}\right)\le\frac{s^2(b+c)^4}{64a^2}$$
$$\Leftrightarrow s^2(s-a)^2-\frac{s^2(s-a)^2(b-c)^2}{a^2}+\frac{s(3s+a).s(s-a)(b-c)^2}{(2s+a)^2}$$
$$-\frac{s(3s+a).s(s-a)(b-c)^4}{a^2(2s+a)^2}\le\frac{s^2(b+c)^4}{64a^2}$$

$$\Leftrightarrow \frac{(2s-a)^4}{64a^2} - (s-a)^2 + (s-a)(b-c)^2\left(\frac{s-a}{a^2} - \frac{3s+a}{(2s+a)^2}\right) + \frac{(3s+a)(s-a)(b-c)^4}{a^2(2s+a)^2} \geq 0$$

$$\Leftrightarrow \frac{\left((2s-a)^2 - 8a(s-a)\right)\left((2s-a)^2 + 8a(s-a)\right)}{64a^2} + \frac{(s-a)(b-c)^2\left((s-a)(2s+a)^2 - a^2(3s+a)\right)}{a^2(2s+a)^2} + \frac{(3s+a)(s-a)(b-c)^4}{a^2(2s+a)^2} \geq 0$$

$$\Leftrightarrow \boxed{\frac{(2s-3a)^2(4s^2+4sa-7a^2)}{64a^2} + \frac{2(s^2-a^2)(2s^2-2sa-a^2)(b-c)^2}{a^2(2s+a)^2} + \frac{(3s+a)(s-a)(b-c)^4}{a^2(2s+a)^2} \overset{(\blacklozenge)}{\geq} 0}$$

$\boxed{\text{Case 1}}$ $(2s-3a)^2 \geq (b-c)^2$ and then : LHS of $(\blacklozenge) \geq$

$$\frac{(4s^2+4sa-7a^2)(b-c)^2}{64a^2} + \frac{2(s^2-a^2)(2s^2-2sa-a^2)(b-c)^2}{a^2(2s+a)^2} \overset{?}{\geq} 0$$

$$\Leftrightarrow \frac{4s^2+4sa-7a^2}{64} + \frac{2(s^2-a^2)(2s^2-2sa-a^2)}{(2s+a)^2} \overset{?}{\geq} 0 \ (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow 272s^4 - 224s^3a - 392s^2a^2 + 232sa^3 + 121a^4 \boxed{\underset{(\blacksquare)}{\overset{?}{\geq}}} 0 \text{ and}$$

$\because 272(s-a)^4 + 864a(s-a)^3 > 0 \therefore$ in order to prove $(\blacksquare)$,
it suffices to prove : LHS of $(\blacksquare) > 272(s-a)^4 + 864a(s-a)^3$
$\Leftrightarrow 568t^2 - 1272t + 713 > 0 \ \left(t = \frac{s}{a}\right)$, which is true $\because$ discriminant
$= 1272^2 - 4(568)(713) = -1952 < 0 \Rightarrow (\blacksquare) \Rightarrow (\blacklozenge)$ is true

$\boxed{\text{Case 2}}$ $(b-c)^2 \geq (2s-3a)^2$ and then : LHS of $(\blacklozenge) \geq$

$$\frac{(4s^2+4sa-7a^2)(2s-3a)^2}{64a^2} + \frac{2(s^2-a^2)(2s^2-2sa-a^2)(b-c)^2}{a^2(2s+a)^2} + \frac{(3s+a)(s-a)(b-c)^2(2s-3a)^2}{a^2(2s+a)^2} = \frac{(4s^2+4sa-7a^2)(2s-3a)^2}{64a^2} + \frac{(s-a)(b-c)^2}{a^2(2s+a)^2}.\left(2(s+a)(2s^2-2sa-a^2) + (3s+a)(2s-3a)^2\right)$$

$$= \frac{(4s^2+4sa-7a^2)(2s-3a)^2}{64a^2} + \frac{(s-a)^2(b-c)^2}{a^2(2s+a)^2}.(16s^2-16sa-7a^2)$$

which is definitely ≥ 0 if : $16s^2-16sa-7a^2 \geq 0$ and so, we now consider :
$16s^2-16sa-7a^2 < 0$ and $\because (b-c)^2 < a^2 \therefore$ LHS of $(\blacklozenge) \geq$

$$\frac{(4s^2+4sa-7a^2)(2s-3a)^2}{64a^2} + \frac{(s-a)^2(b-c)^2}{a^2(2s+a)^2}.(16s^2-16sa-7a^2) > \frac{(4s^2+4sa-7a^2)(2s-3a)^2}{64a^2} + \frac{(s-a)^2(16s^2-16sa-7a^2)}{(2s+a)^2} \overset{?}{>} 0 \Leftrightarrow$$

$$64s^6 - 64s^5a + 752s^4a^2 - 2784s^3a^3 + 2812s^2a^4 - 260sa^5 - 511a^6 \boxed{\underset{(\blacksquare\blacksquare)}{\overset{?}{>}}} 0 \text{ and}$$

$\because (4s-5a)^6 + 104a(s-a)(4s-5a)^4 + 437a^3(4s-5a)^4 > 0$
$\therefore$ in order to prove $(\blacksquare\blacksquare)$, it suffices to prove :
$64 \bullet \text{LHS of } (\blacksquare\blacksquare) > (4s-5a)^6 + 104a(s-a)(4s-5a)^4 + 437a^3(4s-5a)^4$

$$\Leftrightarrow 79232t^3 - 280616t^2 + 329680t - 128227 > 0$$

$$\Leftrightarrow (t-1)\left(79232t^2 - 201384t + 128296\right) + 69 > 0 \to \text{true} \because t = \frac{s}{a} > 1 \text{ and}$$

$$\because \text{discriminant of } \left(79232t^2 - 201384t + 128296\right)$$

$$= 201384^2 - 4(79232)(128296) = -105079232$$

$$\Rightarrow 79232t^2 - 201384t + 128296 > 0 \Rightarrow (\blacksquare\blacksquare) \Rightarrow (\blacklozenge) \text{ is true}$$

$$\therefore \text{combining both cases, } (\blacklozenge) \text{ is true } \forall\, \Delta\, ABC \therefore p_a \le \frac{(b+c)^2}{16r}$$

$$\forall\, \Delta\, ABC, ''='' \text{ iff } 2s - 3a = 0 \text{ and } b = c \Rightarrow ''='' \text{ iff } \Delta\, ABC \text{ is equilateral} \to (3)$$

$$\text{Now, } 4rp_a - rh_a - rr_a \overset{\text{via } (3)}{\le} \frac{(b+c)^2}{4} - r\left(\frac{2rs}{a} + \frac{rs}{s-a}\right)$$

$$= \frac{(b+c)^2}{4} - \frac{r^2 s(b+c-a+a)}{a(s-a)} = \frac{(b+c)^2}{4} - \frac{4(s-b)(s-c)(b+c)}{4a}$$

$$= \frac{(b+c)^2}{4} - \frac{(b+c)\left(a^2-(b-c)^2\right)}{4a} = \frac{a(b+c)^2 - a^2(b+c) + (b+c)(b-c)^2}{4a}$$

$$\overset{?}{<} m_a^2 \Leftrightarrow a(b+c)^2 - a^2(b+c) + (b+c)(b-c)^2 \overset{?}{<} a\left((b+c)^2 - a^2 + (b-c)^2\right)$$

$$\Leftrightarrow a^2(b+c-a) \overset{?}{>} (b+c-a)(b-c)^2 \Leftrightarrow 2(s-a)\left(a^2-(b-c)^2\right) \overset{?}{>} 0$$

$$\Leftrightarrow 8(s-a)(s-b)(s-c) \overset{?}{>} 0 \to \text{true} \therefore 4rp_a - rh_a - rr_a < m_a^2$$

$$\Rightarrow \frac{4p_a - h_a - r_a}{m_a} < \frac{m_a}{r} \text{ and analogs} \therefore \sum_{\text{cyc}} \frac{4p_a - h_a - r_a}{m_a} < \frac{m_a + m_b + m_c}{r}$$

$$\forall\, \Delta\, ABC \text{ (QED)}$$