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In any Δ ABC with p_a, p_b, p_c
→ Spieker cevians, the following relationship holds :

$$1+\sqrt[3]{\prod_{cyc}\frac{n_a+p_a+w_a}{h_a}}\le\frac{2R}{r}$$

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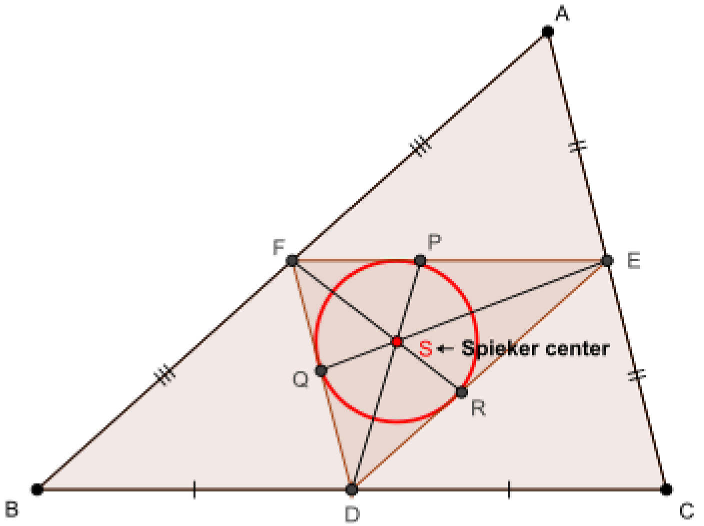

Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF = r′(say)

$$\text{Now, } 16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$$\because \text{Spieker center is incenter of } \Delta DEF, \therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2}\text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS, }\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2} \cdot \frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\blacksquare)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

Also, $p_a^2 - m_a^2 = \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right) - m_a^2$

$$= \frac{2s}{(2s+a)^2}(b^3+c^3-abc) - \left(1 - \frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$= \frac{4(a+b+c)(b^3+c^3-abc) - (2b^2+2c^2-a^2)(b+c)^2}{4(2s+a)^2}$$

$$= \frac{a^2(b-c)^2 + 4a(b+c)(b-c)^2 + 2(b^2-c^2)^2}{4(2s+a)^2}$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\left(\left(a^2+2a(b+c)+(b+c)^2\right) + \left((b+c)^2+2a(b+c)+a^2\right) - a^2\right)$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\left(2(a+b+c)^2 - a^2\right) = \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2 - m_a^2 \overset{(\blacksquare\blacksquare)}{=} \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2} \overset{s>a}{\geq} 0 \Rightarrow p_a \geq m_a \geq w_a \Rightarrow w_a \leq p_a$$

$\therefore$ in order to prove : $\frac{p_a^2-m_a^2}{p_a+m_a} \leq \frac{m_a^2-w_a^2}{m_a+w_a}$, it suffices to prove :

$$p_a^2 - m_a^2 \leq m_a^2 - w_a^2$$

$$\overset{\text{via } (\blacksquare\blacksquare)}{\Leftrightarrow} \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2} \leq s(s-a) + \frac{(b-c)^2}{4} - \left(s(s-a) - \frac{s(s-a)(b-c)^2}{(b+c)^2}\right)$$

$$= \frac{(b-c)^2}{4}\left(1 + \frac{4s(s-a)}{(2s-a)^2}\right) = \frac{(b-c)^2}{4} \cdot \frac{(2s-a)^2+4s(s-a)}{(2s-a)^2}$$

$$\Leftrightarrow \left((2s-a)^2+4s(s-a)\right)(2s+a)^2 \geq (8s^2-a^2)(2s-a)^2$$

$$\Leftrightarrow 16s^3 - 12s^2a - 4sa^2 + a^3 \geq 0 \Leftrightarrow (s-a)(16s^2+4sa) + a^3 \geq 0 \rightarrow \text{true}$$

(strict) since $s > a \therefore \frac{p_a^2-m_a^2}{p_a+m_a} \leq \frac{m_a^2-w_a^2}{m_a+w_a} \Rightarrow p_a + w_a \leq 2m_a$ and analogs $\rightarrow (3)$

Again, Stewart's theorem $\Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$

$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a\left(s^2 - s(2s-a) + bc\right) \Rightarrow s(b^2+c^2) - 2sbc$$

$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2 \Rightarrow an_a^2$$

$$= as^2 + s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$= as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) = as^2 - 2ah_ar_a \therefore n_a^2 = s^2 - 2r_ah_a$$

$$\therefore a^2n_a^2 \overset{?}{\leq} 4(R-r)^2s^2 \Leftrightarrow a^2(s^2-2h_ar_a) \overset{?}{\leq} 4(R-r)^2s^2$$

$$\Leftrightarrow (4R^2\sin^2A)s^2 - 4rs\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(s\tan\frac{A}{2}\right) \overset{?}{\leq} 4(R^2-2Rr+r^2)s^2$$

$$\Leftrightarrow R^2(1-\sin^2A) - 2Rr\left(1 - 2\sin^2\frac{A}{2}\right) + r^2 \overset{?}{\geq} 0 \Leftrightarrow R^2\cos^2A - 2Rr\cos A + r^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (\mathrm{R}\cos\mathrm{A} - \mathrm{r})^2 \overset{?}{\geq} 0 \to \text{true} \therefore an_a \leq 2\mathrm{Rs} - 2\mathrm{rs} \Rightarrow \frac{\mathrm{n}_a}{\mathrm{h}_a} \leq \frac{2\mathrm{Rs}}{a\left(\frac{2\mathrm{rs}}{a}\right)} - \frac{2\mathrm{rs}}{a\left(\frac{2\mathrm{rs}}{a}\right)}$$

$$\Rightarrow \frac{\mathrm{n}_a}{\mathrm{h}_a} \leq \frac{\mathrm{R}}{\mathrm{r}} - 1 \text{ and analogs} \to (4) \therefore (3) \text{ and } (4) \Rightarrow \frac{\mathrm{n}_a + \mathrm{p}_a + \mathrm{w}_a}{\mathrm{h}_a} \leq$$

$$\frac{\mathrm{R}}{\mathrm{r}} - 1 + \frac{2\mathrm{m}_a}{\mathrm{h}_a} \overset{\text{Panaitopol}}{\leq} \frac{\mathrm{R}}{\mathrm{r}} - 1 + \frac{\mathrm{R}}{\mathrm{r}} \therefore \frac{\mathrm{n}_a + \mathrm{p}_a + \mathrm{w}_a}{\mathrm{h}_a} \leq \frac{2\mathrm{R}}{\mathrm{r}} - 1 \text{ and analogs}$$

$$\Rightarrow 1 + \sqrt[3]{\prod_{\text{cyc}} \frac{\mathrm{n}_a + \mathrm{p}_a + \mathrm{w}_a}{\mathrm{h}_a}} \leq 1 + \sqrt[3]{\left(\frac{2\mathrm{R}}{\mathrm{r}} - 1\right)^3} = \frac{2\mathrm{R}}{\mathrm{r}}$$

$$\therefore 1 + \sqrt[3]{\prod_{\text{cyc}} \frac{\mathrm{n}_a + \mathrm{p}_a + \mathrm{w}_a}{\mathrm{h}_a}} \leq \frac{2\mathrm{R}}{\mathrm{r}} \ \forall \Delta \mathrm{ABC}, '' ='' \text{ iff } \Delta \mathrm{ABC} \text{ is equilateral (QED)}$$