ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{a+b-c}{a+c-b}+\frac{a+c-b}{a+b-c}\geq\frac{b}{c}+\frac{c}{b}\ and\ \sqrt{\frac{a+b-c}{a+c-b}}+\sqrt{\frac{a+c-b}{a+b-c}}\geq\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a+b-c}{a+c-b}+\frac{a+c-b}{a+b-c}\geq\frac{b}{c}+\frac{c}{b}\Leftrightarrow\frac{s-b}{s-c}-1+\frac{s-c}{s-b}-1\geq\frac{b}{c}+\frac{c}{b}-2$$

$$\Leftrightarrow\frac{c-b}{s-c}+\frac{b-c}{s-b}\geq\frac{(b-c)^2}{bc}\Leftrightarrow(b-c)\left(\frac{1}{s-b}-\frac{1}{s-c}\right)\geq\frac{(b-c)^2}{bc}$$

$$\Leftrightarrow(b-c)\left(\frac{1}{s-b}-\frac{1}{s-c}\right)\geq\frac{(b-c)^2}{bc}\Leftrightarrow\frac{(b-c)^2}{(s-b)(s-c)}\geq\frac{(b-c)^2}{bc}$$

$$\Leftrightarrow\frac{(b-c)^2}{4bc(s-b)(s-c)}\left(4bc-4(s-b)(s-c)\right)\geq 0$$

$$\Leftrightarrow\frac{(b-c)^2}{4bc(s-b)(s-c)}\left(4bc-a^2+(b-c)^2\right)\geq 0$$

$$\Leftrightarrow\frac{(b-c)^2}{4bc(s-b)(s-c)}\left((b+c)^2-a^2\right)\geq 0\Leftrightarrow\frac{(b-c)^2(b+c+a)(b+c-a)}{4bc(s-b)(s-c)}\geq 0$$

$$\rightarrow \text{true}\ \therefore\frac{a+b-c}{a+c-b}+\frac{a+c-b}{a+b-c}\geq\frac{b}{c}+\frac{c}{b}\ \forall\,\Delta\,ABC$$

$$\text{Let}\ \frac{a+c-b}{a+b-c}=x\ and\ \frac{b}{c}=y\ \therefore\frac{a+b-c}{a+c-b}+\frac{a+c-b}{a+b-c}\geq\frac{b}{c}+\frac{c}{b}$$

$$\Rightarrow x+\frac{1}{x}\geq y+\frac{1}{y}\rightarrow(1)$$

$$\text{So,}\ \sqrt{\frac{a+b-c}{a+c-b}}+\sqrt{\frac{a+c-b}{a+b-c}}\overset{?}{\geq}\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\Leftrightarrow\sqrt{x}+\frac{1}{\sqrt{x}}\overset{?}{\geq}\sqrt{y}+\frac{1}{\sqrt{y}}$$

$$\Leftrightarrow\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2\overset{?}{\geq}\left(\sqrt{y}+\frac{1}{\sqrt{y}}\right)^2\Leftrightarrow x+\frac{1}{x}+2\overset{?}{\geq}y+\frac{1}{y}+2\rightarrow \text{true via}\ (1)$$

$$\therefore\sqrt{\frac{a+b-c}{a+c-b}}+\sqrt{\frac{a+c-b}{a+b-c}}\geq\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\ \forall\,\Delta\,ABC$$

$$\therefore\frac{a+b-c}{a+c-b}+\frac{a+c-b}{a+b-c}\geq\frac{b}{c}+\frac{c}{b}\ and\ \sqrt{\frac{a+b-c}{a+c-b}}+\sqrt{\frac{a+c-b}{a+b-c}}$$

$$\geq\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\ \forall\,\Delta\,ABC,''=''\ \text{iff}\ \Delta\,ABC\ \text{is equilateral (QED)}$$