ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\frac{m_am_bm_c}{sh_ah_bh_c}\cdot\sqrt[4]{\frac{w_aw_bw_c}{p_ap_bp_c}}\geq\frac{\sqrt{xy+yz+zx}}{xh_a+yh_b+zh_c}$$

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We have the following known formula (see [1, pp. 1]),

$$p_a{}^2=s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}.$$

And by the formulas for median and angle bisector of triangle ABC, $m_a{}^2=\frac{1}{4}(2b^2+2c^2-a^2)$

and $w_a=\frac{2\sqrt{bcs(s-a)}}{b+c}$, we can easily get

$$m_a{}^2=s(s-a)+\frac{(b-c)^2}{4}\ \text{ and }\ w_a{}^2=s(s-a)-\frac{s(s-a)(b-c)^2}{(b+c)^2}.$$

Using these identities, we have

$$p_aw_a\leq\frac{p_a{}^2+w_a{}^2}{2}=s(s-a)+\frac{1}{2}\left(\frac{s(3s+a)}{(2s+a)^2}-\frac{s(s-a)}{(2s-a)^2}\right)(b-c)^2$$

$$=s(s-a)+\frac{s(4s^3-4s^2a+sa^2+a^3)(b-c)^2}{(4s^2-a^2)^2}$$

$$=s(s-a)+\left(\frac{1}{4}-\frac{4sa(s-a)(4s+a)+a^4}{4(4s^2-a^2)^2}\right)(b-c)^2\leq m_a{}^2.$$

Then

$$\sqrt[4]{\frac{w_aw_bw_c}{p_ap_bp_c}}=\sqrt[4]{\prod_{cyc}\frac{w_a{}^2}{p_aw_a}}\geq\sqrt[4]{\prod_{cyc}\frac{w_a{}^2}{m_a{}^2}}=\frac{\sqrt{\prod_{cyc}m_aw_a}}{m_am_bm_c}\overset{Panaitopol}{\geq}\frac{\sqrt{\prod_{cyc}s(s-a)}}{m_am_bm_c}=\frac{s^2r}{m_am_bm_c},$$

and since $h_ah_bh_c=\frac{2s^2r^2}{R}$, then we get

$$\frac{m_a m_b m_c}{s h_a h_b h_c} \cdot \sqrt[4]{\frac{w_a w_b w_c}{p_a p_b p_c}} \geq \frac{R}{2s^3 r^2} \cdot s^2 r = \frac{R}{2F}.$$

So it suffices to prove that

$$R(xh_a + yh_b + zh_c) \geq 2F\sqrt{xy + yz + zx} \text{ or } xbc + yca + zab \geq 4F\sqrt{xy + yz + zx}.$$

Let $u := xbc, v := yca, w := zab$. The last inequality is equivalent to

$$u + v + w \geq 4F\sqrt{\frac{uv}{abc^2} + \frac{vw}{a^2bc} + \frac{wu}{ab^2c}}$$

$$\overset{squaring}{\Leftrightarrow} u^2 + v^2 + w^2 \geq 2\left(\frac{8F^2}{abc^2} - 1\right)uv + 2\left(\frac{8F^2}{a^2bc} - 1\right)vw + 2\left(\frac{8F^2}{ab^2c} - 1\right)wu$$

$$\Leftrightarrow u^2 + v^2 + w^2 \geq 2(2\sin A \sin B - 1)uv + 2(2\sin B \sin C - 1)vw + 2(2\sin C \sin A - 1)wu,$$

and since $2\sin B \sin C = \cos A + \cos(B - C) \leq \cos A + 1$, so it suffices to prove that

$$u^2 + v^2 + w^2 \geq 2\cos C \cdot uv + 2\cos A \cdot vw + 2\cos B \cdot wu$$

$$\Leftrightarrow u^2 + v^2 + w^2 \geq 2\cos C \cdot uv + 2(\sin B \sin C - \cos B \cos C) \cdot vw + 2\cos B \cdot wu$$

$$\Leftrightarrow (u - v\cos C - w\cos B)^2 + (v\sin C - w\sin B)^2 \geq 0,$$

which is true and the proof is complete. Equality holds iff ΔABC is equilateral.

[1] Bogdan Fuștei, Mohamed Amine Ben Ajiba,

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