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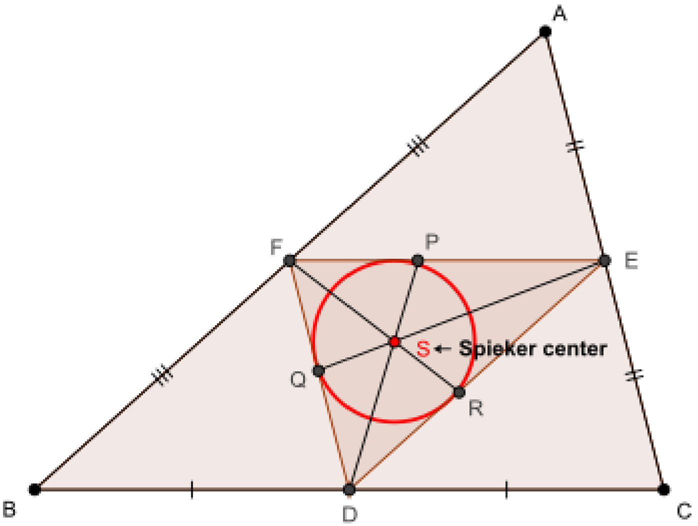
In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\sqrt{(2r_a+h_a)g_a}\ge\sum_{cyc}\left(p_a\cdot\sqrt{\frac{2w_a}{r_b+r_c-4r}}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF = r′(say)

$$\text{Now, } 16[DEF]^2=2\sum_{cyc}\left(\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)\right)-\sum_{cyc}\frac{a^4}{16}=\frac{1}{16}\left(2\sum_{cyc}a^2b^2-\sum_{cyc}a^4\right)$$

$$=\frac{16r^2s^2}{16}\Rightarrow[DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$$\because \text{Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2}\text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\boxed{\overset{(*)}{=}}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\boxed{\overset{(**)}{=}}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\boxed{\overset{(ii)}{=}}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS, }\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$

$$\therefore p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2)=b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}=$$

$$(2s+a).\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}-\frac{a(b-c)^2}{4}$$

$$(a=y+z, b=z+x, c=x+y)$$

$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore (\bullet),(\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$=s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$=s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$=s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)\Rightarrow p_a^2\overset{(\bullet\bullet\bullet)}{=}s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } m_a n_a \overset{?}{\geq} p_a^2+\frac{(b-c)^2}{18}\overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$\left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s(s-a)+\frac{s(b-c)^2}{a}\right)\overset{?}{\geq}\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$

$$+\frac{(b-c)^4}{324}+\frac{(b-c)^2}{9}.\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a}+\frac{1}{4}\right)+\frac{s(b-c)^4}{4a}\overset{?}{\geq}\frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4}+$$

$$2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2}+\frac{(b-c)^4}{324}+s(s-a).\frac{(b-c)^2}{9}+\frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a}+\frac{1}{4}-\frac{2s(3s+a)(b-c)^2}{(2s+a)^2}-\frac{1}{9}\right)+$$

$$\left(\frac{s}{4a}-\frac{s^2(3s+a)^2}{(2s+a)^4}-\frac{1}{324}-\frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2\overset{?}{\geq}0\ (\because (b-c)^2\geq 0)$$

$$\Leftrightarrow \frac{s(s-a)(144s^3-52s^2a-16sa^2+5a^3)}{36a(2s+a)^2}+$$

$$\frac{1296s^5-772s^4a-608s^3a^2+48s^2a^3+37sa^4-a^5}{324a(2s+a)^4}.(b-c)^2\overset{?}{\geq}0$$

$$\Leftrightarrow \frac{s(s-a)\left((s-a)(144s^2+92sa+76a^2)+81a^3\right)}{36a(2s+a)^2}+$$

$$\frac{(s-a)\left((s-a)(1296s^3+1820s^2a+1736sa^2+1700a^3)+1701a^4\right)}{324a(2s+a)^4}.(b-c)^2$$

$$\overset{?}{\geq}0\rightarrow \text{true (strict inequality)}\ \therefore m_a n_a\geq p_a^2+\frac{(b-c)^2}{18}\geq p_a^2\Rightarrow m_a n_a\geq p_a^2\rightarrow \circledast$$

Also, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$ and $b^2(s-b)+c^2(s-c)=ag_a^2+a(s-b)(s-c)$ and via summation, we get :

$$(b^2+c^2)(2s-b-c)=an_a^2+ag_a^2+2a(s-b)(s-c)\Rightarrow 2a(b^2+c^2)=$$
$$2a(n_a^2+g_a^2)+a(a+b-c)(c+a-b)\Rightarrow 2(b^2+c^2)=2(n_a^2+g_a^2)+$$
$$a^2-(b-c)^2\Rightarrow 2(b^2+c^2)-a^2+(b-c)^2=2(n_a^2+g_a^2)\Rightarrow 4m_a^2+(b-c)^2$$
$$=2(n_a^2+g_a^2)\Rightarrow 2(b-c)^2+4s(s-a)=2(n_a^2+g_a^2)$$

$$\Rightarrow n_a^2+g_a^2\ \boxed{\overset{(\bullet)}{=}}\ (b-c)^2+2s(s-a)$$

Again, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+$$
$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)}{bc}$$
$$=as^2-as\left(\frac{a^2-(b-c)^2}{a}\right)\Rightarrow n_a^2=s\left(s-\frac{a^2-(b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2\ \boxed{\overset{(\bullet\bullet)}{=}}\ s(s-a)+\frac{s}{a}(b-c)^2$$

Via $(\bullet)$ and $(\bullet\bullet)$, we get: $g_a^2=(b-c)^2+2s(s-a)-s^2+\dfrac{4s(s-b)(s-c)}{a}$

$$=s^2-2sa+a^2+(b-c)^2-a^2+\frac{4s(s-b)(s-c)}{a}$$
$$=(s-a)^2-4(s-b)(s-c)+\frac{4s(s-b)(s-c)}{a}$$
$$=(s-a)^2+4(s-b)(s-c)\left(\frac{s}{a}-1\right)=(s-a)^2+\frac{4(s-a)(s-b)(s-c)}{a}$$
$$=(s-a)\left(s-a+\frac{a^2-(b-c)^2}{a}\right)\Rightarrow g_a^2\overset{(\bullet\bullet\bullet)}{=}(s-a)\left(s-\frac{(b-c)^2}{a}\right)$$
$$\therefore (\bullet\bullet),(\bullet\bullet\bullet)\Rightarrow n_a^2g_a^2=s(s-a)\left(s-a+\frac{(b-c)^2}{a}\right)\left(s-\frac{(b-c)^2}{a}\right)$$
$$=s(s-a)\left(s(s-a)+s\frac{(b-c)^2}{a}-\frac{(b-c)^2}{a}(s-a)-\frac{(b-c)^4}{a^2}\right)$$

$$\Rightarrow n_a^2 g_a^2 \overset{(\blacksquare)}{=} s(s-a)\left(s(s-a)+(b-c)^2-\frac{(b-c)^4}{a^2}\right)$$

$$\text{Again, } m_a^2 w_a^2 = \frac{(b-c)^2+4s(s-a)}{4}\cdot\frac{4bcs(s-a)}{(b+c)^2}$$

$$\Rightarrow m_a^2 w_a^2 \overset{(\blacksquare\blacksquare)}{=} s(s-a)\frac{bc}{(b+c)^2}\left((b-c)^2+4s(s-a)\right)$$

$$\therefore (\blacksquare),(\blacksquare\blacksquare) \Rightarrow n_a^2 g_a^2 - m_a^2 w_a^2$$

$$= s(s-a)\left(s(s-a)+(b-c)^2-\frac{(b-c)^4}{a^2}-\frac{bc}{(b+c)^2}\left((b-c)^2+4s(s-a)\right)\right)$$

$$= s(s-a)\begin{pmatrix} s(s-a)+(b-c)^2\left(\dfrac{a^2-(b-c)^2}{a^2}\right) \\ -\dfrac{bc}{(b+c)^2}\left((b-c)^2+(b+c)^2-a^2\right)\end{pmatrix}$$

$$= s(s-a)\left(s(s-a)-bc+\left(a^2-(b-c)^2\right)\left(\frac{(b-c)^2}{a^2}+\frac{bc}{(b+c)^2}\right)\right)$$

$$= \frac{s(s-a)}{4}\left(\left((b+c)^2-a^2-4bc\right)+\left(a^2-(b-c)^2\right)\left(\frac{4(b-c)^2}{a^2}+\frac{4bc}{(b+c)^2}\right)\right)$$

$$= \frac{s(s-a)}{4}\left((b-c)^2-a^2+\left(a^2-(b-c)^2\right)\left(\frac{4(b-c)^2}{a^2}+\frac{4bc}{(b+c)^2}\right)\right)$$

$$= \frac{s(s-a)}{4}\left(a^2-(b-c)^2\right)\left(\frac{4(b-c)^2}{a^2}+\frac{4bc}{(b+c)^2}-1\right)$$

$$= \frac{s(s-a)}{4}.4(s-b)(s-c)\left(\frac{4(b-c)^2}{a^2}-\frac{(b-c)^2}{(b+c)^2}\right) = r^2s^2(b-c)^2\left(\frac{4}{a^2}-\frac{1}{(b+c)^2}\right)$$

$$= r^2s^2(b-c)^2\left(\frac{2}{a}+\frac{1}{b+c}\right)\left(\frac{2b+2c-a}{a(b+c)}\right) \ge 0$$

$$\Rightarrow n_a^2 g_a^2 \ge m_a^2 w_a^2 \Rightarrow n_a g_a \ge m_a w_a \to \circledast\circledast$$

$$\text{We have}: (2r_a+h_a)(r_b+r_c-4r) = \left(\frac{2rs}{s-a}+\frac{2rs}{a}\right)\left(\frac{rs}{s-b}+\frac{rs}{s-c}-\frac{4rs}{s}\right)$$

$$= \frac{2r^2s^2\left(s(s-c)+s(s-b)-4(s-b)(s-c)\right)}{a(s-a)(s-b)(s-c)}$$

$$= \frac{2r^2s^2}{a.r^2s}\left(s(2s-2s+a)-a^2+(b-c)^2\right) = 2\left(s(s-a)+\frac{s}{a}(b-c)^2\right) \overset{\text{via }(\bullet\bullet)}{=} 2n_a^2$$

$$\Rightarrow (2r_a+h_a)(r_b+r_c-4r).\frac{g_a}{p_a^2.2w_a} = \frac{n_a^2 g_a}{p_a^2 w_a} \overset{\text{via }\circledast\circledast}{\ge} \frac{n_a m_a w_a}{p_a^2 w_a} \overset{\text{via }\circledast\circledast}{\ge} 1$$

$$\Rightarrow \sqrt{(2r_a+h_a)(r_b+r_c-4r).g_a} \ge p_a.\sqrt{2w_a}$$

$$\Rightarrow \sqrt{(2r_a+h_a)g_a} \ge p_a.\sqrt{\frac{2w_a}{r_b+r_c-4r}} \text{ and analogs}$$

$$\Rightarrow \sum_{cyc}\sqrt{(2r_a+h_a)g_a} \ge \sum_{cyc}\left(p_a.\sqrt{\frac{2w_a}{r_b+r_c-4r}}\right) \forall \Delta ABC,$$

$'' = ''$ iff Δ ABC is equilateral (QED)