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In ΔABC the following relationship holds:

$$2\left(\frac{2m_bm_c}{h_bh_c}-1\right)\left(\frac{r_b}{r_c}+\frac{r_c}{r_b}\right)\geq\left(\frac{b}{c}+\frac{c}{b}\right)^2$$

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By Tereshin and CBS inequalities, we have

$$m_bm_c\geq\frac{(c^2+a^2)(a^2+b^2)}{(4R)^2}\geq\frac{(ca+ab)^2}{(4R)^2}=\frac{a^2(b+c)^2}{(4R)^2}.$$

and since $h_a=\frac{bc}{2R}$ (and analogs), then

$$\frac{2m_bm_c}{h_bh_c}-1\geq\frac{a^2(b+c)^2}{8R^2}\cdot\frac{4R^2}{a^2bc}-1=\frac{(b+c)^2}{2bc}-1=\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right).$$

Now, we have

$$\left(\frac{r_b}{r_c}+\frac{r_c}{r_b}\right)-\left(\frac{b}{c}+\frac{c}{b}\right)=\left(\frac{s-c}{s-b}-\frac{c}{b}\right)-\left(\frac{b}{c}-\frac{s-b}{s-c}\right)=\frac{s(b-c)}{b(s-b)}-\frac{s(b-c)}{c(s-c)}=\frac{s(s-a)(b-c)^2}{bc(s-b)(s-c)}\geq 0,$$

$$\Rightarrow\frac{r_b}{r_c}+\frac{r_c}{r_b}\geq\frac{b}{c}+\frac{c}{b}.$$

Therefore

$$2\left(\frac{2m_bm_c}{h_bh_c}-1\right)\left(\frac{r_b}{r_c}+\frac{r_c}{r_b}\right)\geq\left(\frac{b}{c}+\frac{c}{b}\right)^2.$$

Equality holds iff ΔABC is equilateral.