ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$(n_a+n_b+n_c)^2 \le (8R+2r+h_a+h_b+h_c)(4R-5r)$$

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$$\text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b) = an_a^2 + a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a)+bc) \Rightarrow s(b^2+c^2) - 2sbc$$
$$= an_a^2 + a(as-s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 +$$
$$s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$= as^2 - a.\frac{4r^2s^2}{a(s-a)} = as^2 - 2a.\frac{2rs}{a}.\frac{rs}{s-a} = as^2 - 2ah_ar_a \Rightarrow n_a^2 = s^2 - 2h_ar_a$$
$$= s^2 - \frac{4rs.s\tan\frac{A}{2}}{4R\cos^2\frac{A}{2}\tan\frac{A}{2}} \Rightarrow n_a^2 = s^2 - \frac{rs^2}{R}.\sec^2\frac{A}{2} \text{ and analogs}$$
$$\Rightarrow \sum_{cyc} n_a^2 = 3s^2 - \frac{rs^2}{R}.\frac{s^2+(4R+r)^2}{s^2} \Rightarrow (n_a+n_b+n_c)^2 \le 3\sum_{cyc} n_a^2$$
$$= 3.\frac{(3R-r)s^2 - r(4R+r)^2}{R} \overset{?}{\le} (8R+2r+h_a+h_b+h_c)(4R-5r)$$
$$\Leftrightarrow 6\left((3R-r)s^2 - r(4R+r)^2\right) - (4R-5r)(s^2+4Rr+r^2)$$
$$\overset{?}{\le} 2R(8R+2r)(4R-5r) \Leftrightarrow$$
$$(14R-r)s^2 \underset{(*)}{\overset{?}{\le}} 2R(8R+2r)(4R-5r) + 6r(4R+r)^2 + (4R-5r)(4Rr+r^2)$$
$$\text{Now, } (14R-r)s^2 \overset{\text{Gerretsen}}{\le} (14R-r)(4R^2+4Rr+3r^2) \overset{?}{\le} \text{RHS of } (*)$$
$$\Leftrightarrow 4t^3 - 2t^2 - 13t + 2 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(4t^2+6t-1) \overset{?}{\ge} 0 \to \text{true}$$
$$\because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (*) \text{ is true } \therefore (n_a+n_b+n_c)^2 \le (8R+2r+h_a+h_b+h_c)(4R-5r)$$
$$\forall \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$