ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationships hold :

$$2\sum_{cyc}\frac{\sqrt{m_a}}{a} \geq \sum_{cyc}\sqrt{\frac{w_a}{bc-w_a^2}} \text{ and } 4\sum_{cyc}\frac{m_a}{r_a-r} \geq \sum_{cyc}\frac{w_a(r_b+r_c)}{bc-w_a^2}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$bc-w_a^2 = bc-\frac{4bcs(s-a)}{(b+c)^2} = bc-bc.\frac{(b+c)^2-a^2}{(b+c)^2}$$

$$\Rightarrow bc-w_a^2 = \frac{a^2bc}{(b+c)^2} \rightarrow (1)$$

$$\text{Also, } r_b+r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c = 4R\cos^2\frac{A}{2} \rightarrow (2)$$

$$\text{Now, } \frac{2\sqrt{m_a}}{a}.\sqrt{\frac{bc-w_a^2}{w_a}} \overset{\text{via (1) and Lascu}}{\geq} \frac{2\sqrt{\frac{b+c}{2}\cos\frac{A}{2}}}{a}.\sqrt{\frac{\frac{a^2bc}{(b+c)^2}}{\frac{2bc}{b+c}\cos\frac{A}{2}}} = 1$$

$$\Rightarrow \frac{2\sqrt{m_a}}{a} \geq \sqrt{\frac{w_a}{bc-w_a^2}} \text{ and analogs } \therefore 2\sum_{cyc}\frac{\sqrt{m_a}}{a} \geq \sum_{cyc}\sqrt{\frac{w_a}{bc-w_a^2}}$$

$$\text{Again, } \frac{4m_a}{r_a-r}.\frac{bc-w_a^2}{w_a(r_b+r_c)} \overset{\text{via (1),(2) and Lascu}}{\geq} \frac{4.\frac{b+c}{2}\cos\frac{A}{2}}{\frac{rs(s-(s-a))}{s(s-a)}}.\frac{\frac{a^2bc}{(b+c)^2}}{\frac{2bc}{b+c}\cos\frac{A}{2}.4R\frac{s(s-a)}{bc}} = \frac{abc}{4Rrs}$$

$$= 1 \Rightarrow \frac{4m_a}{r_a-r} \geq \frac{w_a(r_b+r_c)}{bc-w_a^2} \text{ and analogs } \therefore 4\sum_{cyc}\frac{m_a}{r_a-r} \geq \sum_{cyc}\frac{w_a(r_b+r_c)}{bc-w_a^2} \text{ (QED)}$$