ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\cdot\frac{a}{\sqrt{bc-w_a^2}}\le\sqrt{\frac{2R}{r}}$$

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$$bc-w_a^2=bc-\frac{4bcs(s-a)}{(b+c)^2}=bc-bc.\frac{(b+c)^2-a^2}{(b+c)^2}=\frac{a^2bc}{(b+c)^2}$$

$$\Rightarrow\frac{a}{\sqrt{bc-w_a^2}}=\frac{b+c}{\sqrt{bc}}\therefore\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\cdot\frac{a}{\sqrt{bc-w_a^2}}\le\sqrt{\frac{2R}{r}}$$

$$\Leftrightarrow\frac{2R(4R+r)}{ab+bc+ca}\cdot\frac{(b+c)^2}{bc}\le\frac{2R}{r}\Leftrightarrow\frac{4R+r}{ab+bc+ca}\cdot\frac{a(b+c)^2}{4Rrs}\le\frac{1}{r}$$

$$\Leftrightarrow\frac{4R+r}{4R}\le\frac{s(ab+bc+ca)}{a(b+c)^2}\Leftrightarrow\frac{r}{4R}\le\frac{s(ab+bc+ca)-a(b+c)^2}{a(b+c)^2}$$

$$\Leftrightarrow\frac{F^2}{sabc}\le\frac{s(ab+bc+ca)-a(b+c)^2}{a(b+c)^2}$$

$$\Leftrightarrow(s-a)(s-b)(s-c)(b+c)^2\le bc\big(s(ab+bc+ca)-a(b+c)^2\big)$$

$$\Leftrightarrow(z+x)(x+y)\left(\begin{matrix}\left(\sum_{cyc}x\right)\left(\sum_{cyc}((z+x)(x+y))\right)\\-(y+z)(2x+y+z)^2\end{matrix}\right)-xyz(2x+y+z)^2\ge0$$

$$(\text{where } x=s-a,y=s-b,z=s-c\Rightarrow a=y+z,b=z+x,c=x+y)$$

$$\Leftrightarrow x^5+x^4y+x^4z+xy^2z^2+y^3z^2+y^2z^3\ge2x^3yz+2x^2y^2z+2x^2yz^2\to\text{true}$$

$$\because x^5+xy^2z^2\overset{A-G}{\ge}2x^3yz,x^4y+y^3z^2\overset{A-G}{\ge}2x^2y^2z\text{ and }x^4z+y^2z^3\overset{A-G}{\ge}2x^2yz^2$$

$$\therefore\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\cdot\frac{a}{\sqrt{bc-w_a^2}}\le\sqrt{\frac{2R}{r}}\ \forall\,\Delta\,ABC\ (QED)$$