ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds :

$$\frac{n_a n_b n_c}{r_a r_b r_c} \geq \frac{\sqrt{3}}{R}\left(max(a,b,c) - min(a,b,c)\right)$$

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We assume that $a \geq b \geq c$. Since we have

$$(a-c)^2 = [(a-b)+(b-c)]^2 \overset{CBS}{\leq} 2\left[(a-b)^2 + (b-c)^2\right],$$

then

$$\sqrt{3}\left(max(a,b,c) - min(a,b,c)\right) = \sqrt{3(a-c)^2} \leq \sqrt{2[(a-b)^2+(b-c)^2+(a-c)^2]}$$

$$= 2\sqrt{s^2 - 3r^2 - 12Rr}.$$

So it suffices to prove that

$$Rn_a n_b n_c \geq 2s^2 r\sqrt{s^2 - 3r^2 - 12Rr} \qquad (*)$$

We have

$$n_a{}^2 = s(s-a) + \frac{s(b-c)^2}{a} = s^2 - \frac{s\left[a^2 - (b-c)^2\right]}{a} = s^2 - \frac{4s(s-b)(s-c)}{a}$$

$$= s^2 - \frac{4s.sr^2}{a(s-a)} = s^2 - 2h_a r_a, \text{ then}$$

$$(n_a n_b n_c)^2 = \left(s^2 - 2h_a r_a\right)\left(s^2 - 2h_b r_b\right)\left(s^2 - 2h_c r_c\right)$$

$$= s^6 - 2s^4 \sum_{cyc} h_a r_a + 4r_a r_b r_c h_a h_b h_c \left(s^2 \sum_{cyc} \frac{1}{h_a r_a} - 2\right)$$

$$= s^6 - 2s^4.\frac{r\left[s^2 + (4R+r)^2\right]}{2R} + 4s^2 r.\frac{2s^2 r^2}{R}\left(s^2.\frac{4R+r}{s^2 r} - 2\right) = \frac{s^4\left[(R-r)s^2 - r(4R-3r)^2\right]}{R}.$$

So the inequality $(*)$ is equivalent to

$$R\left[(R-r)s^2 - r(4R-3r)^2\right] \geq 4r^2\left(s^2 - 3r^2 - 12Rr\right)$$

$$\Leftrightarrow \left(R^2 - Rr - 4r^2\right)s^2 \geq 16R^3 r - 24R^2 r^2 - 39Rr^3 - 12r^4 \qquad (1)$$

▪*If* $R^2 - Rr - 4r^2 \geq 0$, we have

$$LHS_{(1)} \overset{Rouche}{\geq} \left(R^2 - Rr - 4r^2\right)\left(2R^2 + 10Rr - r^2 - 2\sqrt{R(R-2r)^3}\right)$$

$= 2R^4 + 8R^3r - 19R^2r^2 - 39Rr^3 + 4r^4 - 2(R^2 - Rr - 4r^2)\sqrt{R(R-2r)^3}$

$\overset{AM-GM}{\overbrace{\geq}} \; 2R^4 + 8R^3r - 19R^2r^2 - 39Rr^3 + 4r^4 - \left[(R^2 - Rr - 4r^2)^2 + R(R-2r)^3\right] = RHS_{(1)}.$

▪$If\ R^2 - Rr - 4r^2 \leq 0$, **we have**

$LHS_{(1)} \overset{Rouche}{\overbrace{\geq}} (R^2 - Rr - 4r^2)\left(2R^2 + 10Rr - r^2 + 2\sqrt{R(R-2r)^3}\right)$

$= 2R^4 + 8R^3r - 19R^2r^2 - 39Rr^3 + 4r^4 - 2(Rr + 4r^2 - R^2)\sqrt{R(R-2r)^3}$

$\overset{AM-GM}{\overbrace{\geq}} \; 2R^4 + 8R^3r - 19R^2r^2 - 39Rr^3 + 4r^4 - \left[(Rr + 4r^2 - R^2)^2 + R(R-2r)^3\right] = RHS_{(1)}.$

which completes the proof.