ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{2m_a w_a}{h_a} \geq r_b + r_c \geq \frac{m_b h_b + m_c h_c}{r_a}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2} \therefore r_b + r_c \overset{(1)}{=} 4R\cos^2\frac{A}{2}$$

$$\text{Now, } r_b + r_c - \frac{m_b h_b + m_c h_c}{r_a} \geq r_b + r_c - \frac{as}{r_a}$$

$$\left(\begin{gathered}\frac{R}{r} \geq \frac{m_b}{h_c} + \frac{m_c}{h_b} \ldots \text{reference : article titled} \\ \text{"New Triangle Inequalities With Brocard's Angle"} \\ \text{by Bogdan Fustei, Mohamed Amine Ben Ajiba; Lemma 12, 6 − 7,} \\ \text{published at : www.ssmrmh.ro} \therefore m_b h_b + m_c h_c \leq \frac{R}{r}.h_b h_c = \frac{R}{r}.\frac{4r^2s^2.a}{bca} \\ = \frac{R}{r}.\frac{4r^2s^2.a}{4Rrs} \therefore m_b h_b + m_c h_c \leq as\end{gathered}\right)$$

$$= \frac{\sum_{cyc} r_a r_b - r_b r_c - as}{r_a} = \frac{s^2 - s(s-a) - as}{r_a} = 0 \Rightarrow r_b + r_c - \frac{m_b h_b + m_c h_c}{r_a} \geq 0$$

$$\therefore \boxed{r_b + r_c \geq \frac{m_b h_b + m_c h_c}{r_a}}$$

$$\text{Again, } \frac{2m_a w_a}{h_a} - (r_b + r_c) \overset{\text{Lascu}}{\geq} \frac{2.\frac{b+c}{2}.\cos\frac{A}{2}.\frac{2bc}{b+c}.\cos\frac{A}{2}}{\frac{bc}{2R}} - (r_b + r_c) \overset{\text{via (1)}}{=}$$

$$4R\cos^2\frac{A}{2} - 4R\cos^2\frac{A}{2} = 0 \therefore \boxed{\frac{2m_a w_a}{h_a} \geq r_b + r_c}$$

$$\therefore \frac{2m_a w_a}{h_a} \geq r_b + r_c \geq \frac{m_b h_b + m_c h_c}{r_a} \text{ (QED)}$$