ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$(4R + r_a)(r_b + r_c - h_a) \geq n_a^2 + g_a^2 + m_b h_b + m_c h_c - r_a(h_a - 2r)$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b + r_c \stackrel{(1)}{=} 4R\cos^2\frac{A}{2}$$

$$\text{Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) \stackrel{(i)}{=} a n_a^2 + a(s-b)(s-c) \text{ and}$$

$$b^2(s-b) + c^2(s-c) \stackrel{(ii)}{=} a g_a^2 + a(s-b)(s-c) \text{ and (i)} + \text{(i)} \Rightarrow$$

$$(b^2 + c^2)(2s - b - c) = a n_a^2 + a g_a^2 + 2a(s-b)(s-c)$$

$$\Rightarrow 2a(b^2 + c^2) = 2a(n_a^2 + g_a^2) + a(a + b - c)(c + a - b)$$

$$\Rightarrow 2(b^2 + c^2) = 2(n_a^2 + g_a^2) + a^2 - (b-c)^2 \Rightarrow 2(b^2 + c^2) - a^2 + (b-c)^2 =$$

$$2\left(n_a^2 + g_a^2\right) \Rightarrow 4m_a^2 + (b-c)^2 = 2\left(n_a^2 + g_a^2\right) \Rightarrow 2\left(n_a^2 + g_a^2\right)$$

$$= (b-c)^2 + 4s(s-a) + (b-c)^2 \therefore n_a^2 + g_a^2 \stackrel{(*)}{=} (b-c)^2 + 2s(s-a)$$

$$\therefore (4R + r_a)(r_b + r_c - h_a) - \left(n_a^2 + g_a^2 + m_b h_b + m_c h_c - r_a(h_a - 2r)\right)$$

$$\stackrel{\text{via }(*)}{=} 4R(r_b + r_c) - 4R.\frac{bc}{2R} + r_a(r_b + r_c) - r_a h_a - (b-c)^2 - 2s(s-a)$$

$$-(m_b h_b + m_c h_c) + r_a h_a - 2r r_a \geq$$

$$4R(r_b + r_c) - 2bc + r_a(r_b + r_c) - (b-c)^2$$

$$-\frac{1}{2}\left((b+c)^2 - a^2 + \frac{4(s-a)(s-b)(s-c)}{(s-a)}\right) - as$$

$$\left(\begin{array}{c}\frac{R}{r} \geq \frac{m_b}{h_c} + \frac{m_c}{h_b} \ldots \text{reference : article titled} \\ \text{"New Triangle Inequalities With Brocard's Angle"} \\ \text{by Bogdan Fustei, Mohamed Amine Ben Ajiba; Lemma 12, 6 - 7,} \\ \text{published at : www.ssmrmh.ro} \therefore m_b h_b + m_c h_c \leq \frac{R}{r}.h_b h_c = \frac{R}{r}.\frac{4r^2 s^2.a}{bca} \\ = \frac{R}{r}.\frac{4r^2 s^2.a}{4Rrs} \therefore m_b h_b + m_c h_c \leq as\end{array}\right)$$

$$= 4R(r_b + r_c) + r_a(r_b + r_c) - \left(b^2 + c^2\right) - \frac{1}{2}\left((b+c)^2 - a^2 + a^2 - (b-c)^2\right) - as$$

$$\stackrel{\text{via }(1)}{=} 4R.4R\cos^2\frac{A}{2} + \sum_{\text{cyc}} r_a r_b - r_b r_c - \left(b^2 + c^2 + 2bc\right) - as$$

$$= 16R^2\cos^2\frac{A}{2} + s^2 - s(s-a) - (b+c)^2 - as$$

$$= 16R^2\cos^2\frac{A}{2} - 16R^2\cos^2\frac{A}{2}\cos^2\frac{B-C}{2} = 16R^2\cos^2\frac{A}{2}\left(1 - \cos^2\frac{B-C}{2}\right) \geq 0$$

$$\therefore (4R + r_a)(r_b + r_c - h_a) \geq n_a^2 + g_a^2 + m_b h_b + m_c h_c - r_a(h_a - 2r) \text{ (QED)}$$