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In ΔABC the following relationship holds:

$$\left(\sum_{cyc}\sqrt[5]{\sin A}\right)\left(\sum_{cyc}\sqrt[5]{\sin\frac{A}{2}}\right)\le\sqrt[10]{\frac{3^{21}}{16}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by George Florin Serban

$$f:(0,\pi)\to(0,1],f(x)=\sqrt[5]{\sin x}=(\sin x)^{\frac{1}{5}}$$

$$f'(x)=\frac{1}{5}(\sin x)^{-\frac{4}{5}}\cdot\cos x\,,f''(x)=-\frac{4}{25}(\sin x)^{-\frac{9}{5}}\cos^2 x-\frac{1}{5}(\sin x)^{-\frac{4}{5}}\sin x<0$$

$$(\forall)x\in(0,\pi)\Rightarrow f\text{ concave}$$

$$f\left(\frac{A+B+C}{3}\right)\ge\frac{\sum_{cyc}f(A)}{3}\overset{Jensen}{\Rightarrow}\sum_{cyc}f(A)\le 3f(60^\circ)\Rightarrow\sum_{cyc}\sqrt[5]{\sin A}\le 3\left(\frac{\sqrt{3}}{2}\right)^{\frac{1}{5}}$$

$$g:(0,\pi)\to(0,1),g(x)=\sqrt[5]{\sin\frac{x}{2}}=\left(\sin\frac{x}{2}\right)^{\frac{1}{5}}$$

$$g'(x)=\frac{1}{5}\left(\sin\frac{x}{2}\right)^{-\frac{4}{5}}\frac{1}{2}\cos\frac{x}{2}=\frac{1}{10}\left(\sin\frac{x}{2}\right)^{-\frac{4}{5}}\cos\frac{x}{2}$$

$$g''(x)=-\frac{4}{50}\left(\sin\frac{x}{2}\right)^{-\frac{9}{5}}\cdot\frac{1}{2}\cdot\cos^2\frac{x}{2}-\frac{1}{10}\left(\sin\frac{x}{2}\right)^{-\frac{4}{5}}\cdot\frac{1}{2}\sin\frac{x}{2}$$

$$g''(x)<0,(\forall)x\in(0,\pi)\Rightarrow g\text{ concave}\to\text{Jensen}$$

$$g\left(\frac{A+B+C}{3}\right)\ge\frac{\sum_{cyc}g(A)}{3}\Rightarrow\sum_{cyc}g(A)\le 3g(60^\circ)\Rightarrow\sum_{cyc}\sqrt[5]{\sin\frac{A}{2}}\le 3\left(\frac{1}{2}\right)^{\frac{1}{5}}$$

$$\Rightarrow\left(\sum_{cyc}\sqrt[5]{\sin A}\right)\cdot\left(\sum_{cyc}\sqrt[5]{\sin\frac{A}{2}}\right)\le 9\left(\frac{\sqrt{3}}{4}\right)^{\frac{1}{5}}=\frac{3^{2+\frac{1}{10}}}{4^{\frac{1}{5}}}=$$

$$=\frac{3^{\frac{21}{10}}}{2^{\frac{2}{5}}}=\frac{\sqrt[10]{3^{21}}}{2^{\frac{4}{10}}}=\frac{\sqrt[10]{3^{21}}}{\sqrt[10]{2^4}}=\sqrt[10]{\frac{3^{21}}{16}}$$

$$\Rightarrow \left(\sum_{cyc}\sqrt[5]{\sin A}\right)\cdot\left(\sum_{cyc}\sqrt[5]{\sin\frac{A}{2}}\right)\le\sqrt[10]{\frac{3^{21}}{16}}$$

Solution 2 by Tapas Das-India

$$\sin^2 A+\sin^2 B+\sin^2 C=\frac{a^2}{4R^2}+\frac{b^2}{4R^2}+\frac{c^2}{4R^2}=\frac{a^2+b^2+c^2}{4R^2}$$
$$\le\frac{9R^2}{4R^2}=\frac{9}{4}$$

$$\sum\sqrt[5]{\sin A}\overset{Leibniz}{\le}\sum\sqrt[10]{\sin^2 A}\overset{CBS}{\le}\frac{3}{3^{10}}\left(\sum\sin^2 A\right)^{\frac{1}{10}}\le\frac{3}{3^{10}}\cdot\left(\frac{9}{4}\right)^{\frac{1}{10}}$$

Let $f(x)=\sin\frac{x}{2}, x\in(0,\pi)$

$$f'(x)=\frac{1}{2}\cos\frac{x}{2}$$

$$f''(x)=-\frac{1}{4}\sin^2\frac{x}{2}<0$$

$\therefore f$ is concave

$$f\left(\frac{A}{2}\right)+f\left(\frac{B}{2}\right)+f\left(\frac{C}{2}\right)\le 3f\left(\frac{A+B+C}{6}\right)$$

$$\therefore\sum\sin\frac{A}{2}=3\cdot\sin\frac{\pi}{6}=\frac{3}{2}$$

$$\sum\sqrt[5]{\sin\frac{A}{2}}\overset{CBS}{\le}\frac{3}{3^{\frac{1}{5}}}\left(\sum\left(\sin\frac{A}{2}\right)\right)^{\frac{1}{5}}$$

$$=\frac{3}{3^{\frac{1}{5}}}\left(\frac{3}{2}\right)^{\frac{1}{5}}=\frac{3}{2^{\frac{1}{5}}}$$

$$\sum\sqrt[5]{\sin A}\cdot\sum\sqrt[5]{\sin\frac{A}{2}}=\frac{(3^{10})^{\frac{1}{10}}}{(2^2)^{\frac{1}{10}}}$$

$$\le\frac{3}{3^{10}}\left(\frac{9}{4}\right)^{\frac{1}{10}}\cdot\frac{(3^{10})^{\frac{1}{10}}}{(2^2)^{\frac{1}{10}}}=\left(\frac{3^{10}\cdot 3^2\cdot 3^{10}}{3\cdot 2^2\cdot 2^2}\right)^{\frac{1}{10}}=\left(\frac{3^{21}}{16}\right)^{\frac{1}{10}}$$

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Solution 3 by Khaled Abd Imouti-Syria

Let be the function:

$$f(x)=\sqrt[5]{\sin x}, x\in]0,\pi[$$

$$f(x)=(\sin x)^{\frac{1}{5}}, 'f(x)=\frac{1}{5}(\sin x)^{-\frac{4}{5}}\cdot\cos x$$

$$''f(x)=\frac{1}{5}\left[\underbrace{-\frac{4}{5}(\sin x)^{-\frac{9}{5}}\cos^2 x}_{<0}-\underbrace{\sin x\cdot(\sin x)^{-\frac{4}{5}}}_{<0}\right]<0$$

So f is a concave function

$$\sum_{cyc}\sqrt[5]{\sin A}\le\sqrt[5]{\sin\left(\frac{A+B+C}{3}\right)}, \sum_{cyc}\sqrt[5]{\sin A}\le 3\sqrt[5]{\frac{\sqrt{3}}{2}}=3\frac{(3)^{\frac{1}{10}}}{(2)^{\frac{1}{5}}}\quad (*)$$

Let be the function: $g(x)=\sqrt[5]{\sin\frac{x}{2}}, x\in]0,\pi[$

in similar way g is concave function

$$\text{So: } \sum_{cyc}\sqrt[5]{\sin\frac{A}{2}}\le 3\sqrt[5]{\sin\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right)}\to\sum_{cyc}\sqrt[5]{\sin\frac{A}{2}}\le 3\sqrt[5]{\frac{1}{2}}\quad (**)$$

$\Rightarrow$ from (*) and (**)

$$\left(\sum_{cyc}\sqrt[5]{\sin A}\right)\cdot\left(\sum_{cyc}\sqrt[5]{\sin\frac{A}{2}}\right)\le 9\cdot\frac{(3)^{\frac{1}{10}}}{(2)^{\frac{2}{5}}}=\frac{(3)^2\cdot(3)^{\frac{1}{10}}}{(4)^{\frac{1}{5}}}=\frac{(3)^{\frac{21}{10}}}{(4^2)^{\frac{1}{10}}}=\sqrt[10]{\frac{3^{21}}{16}}$$