ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$a^8+b^8+c^8\ge \frac{256}{3}\cdot F^4+\frac{1}{2}\sum_{cyc}(a^4-b^4)^2$$

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We know that

$$16F^2=2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4) \qquad \textbf{(1)}$$

Since, $(b^2-c^2)^2+(c^2-a^2)^2+(a^2-b^2)^2\ge 0$

i.e., $b^2c^2+c^2a^2+a^2b^2\le a^4+b^4+c^4$

So, we can say from (1)

$$16F^2\le b^2c^2+c^2a^2+a^2b^2$$

$$\therefore F^4\le \frac{1}{256}(a^2b^2+b^2c^2+c^2a^2)^2 \qquad \textbf{(2)}$$

$$\frac{1}{2}\left[\sum(a^4-b^4)^2\right]=\frac{1}{2}\left[2\left(\sum a^8\right)-2\left(\sum a^4b^4\right)\right]=\left(\sum a^8\right)-\sum a^4b^4$$

We need to show

$$a^8+b^8+c^8\ge \frac{256}{3}F^4+\frac{1}{2}\sum(a^4-b^4)^2$$

Or, $a^8+b^8+c^8\ge \frac{256}{3}F^4+\left(\sum a^8\right)-\sum a^4b^4$

Or $\sum a^4b^4\ge \frac{256}{3}F^4$

Or, $\sum a^4b^4\ge \frac{256}{3}\cdot\frac{1}{256}(b^2c^2+c^2a^2+a^2b^2)^2$

[Using relation (2)]

Or $\sum a^4b^4\ge \frac{1}{3}(b^2c^2+c^2a^2+a^2b^2)^2$

This is true by CBS inequality.